The entire exam deals with unity-feedback control. The reference input is \( r(t) \) and the control output is \( y(t) \). The plant transfer function is \( G(s) \) and the controller transfer function is \( D(s) \). The system block diagram is shown below.

\[
\begin{align*}
 r(t) & \quad \rightarrow \quad D(s) \quad \rightarrow \quad G(s) \quad \rightarrow \quad y(t)
\end{align*}
\]

The plant transfer function is given by:

\[
G(s) = \frac{s + 10}{s(s + 1)}
\]

For all problems (1 through 3), proportional control is used so that \( D(s) = k_p \).

1. Closed-loop transfer function.

A. (1 pt.) Find the closed-loop transfer function, \( T(s) = \frac{Y(s)}{R(s)} \).

B. (2 pts.) For the closed-loop system, find the undamped natural frequency, \( \omega_n \), and the damping ratio, \( \zeta \), as a function of the proportional gain, \( k_p \).

2. Root Locus. The root locus plot from MATLAB is shown on the next page. Note that there is a breakaway point at A and a break-in point at C. Detailed analysis shows that the points A, B, B' and C lie on a circle.

A. (2 pts.) Calculate the location in the s-plane of points A, B, B' and C.

B. (2 pts.) Find the values of the gain, \( k_p \), for the closed-loop poles to lie at point A, points B and B', and point C.

C. (2 pts.) For each closed-loop pole location (A, B and B' or C) estimate the rise time, \( t_r \).

D. (1 pt.) Estimate the overshoot, \( M_p \), when the closed-loop poles are at B and B'.

E. (2 pts.) For estimates of \( t_r \) and \( M_p \) from parts C and D, do you expect the exact values to be lower or higher than the estimates? Briefly explain your reasoning.
3. Frequency Response.

A. (2 pts.) Find expressions for the magnitude, $M(\omega)$, and the phase, $\phi(\omega)$, of the frequency response for the open-loop transfer function, $D(s)G(s)$.

B. (3 pts.) Sketch the Bode plot (log magnitude and phase versus log frequency) of the open-loop transfer function with $k_p = 1$. Clearly indicate the slope of any asymptote lines and any corner frequencies on the Bode plot.

C. (3 pts.) For $k_p=1$, what is the crossover frequency ($\omega_C$) and phase margin (PM)? Estimate the damping ratio from the PM and compare the result to the exact value from Problem 1.
1. Closed-loop transfer function.

A. \[ T(s) = \frac{Y(s)}{R(s)} = \frac{D(s) G(s)}{1 + D(s) G(s)} \]

\[ T(s) = \frac{k_p (s+10)}{k_p (s+10) + s(s+1)} = \frac{k_p (s+10)}{s^2 + (1+k_p)s + 10k_p} \]

B. Characteristic equation:

\[ s^2 + (1+k_p)s + 10k_p = 0 \]

\[ \omega_n = \sqrt{10k_p} \]

\[ s = \frac{1+k_p}{2\omega_n} = \frac{1+k_p}{2\sqrt{10k_p}} \]
2. Root locus

\[ A(s) = D(s) G(s) = \frac{b(s)}{K} \]

\[ \frac{s+10}{s(s+1)} \]

\[ \text{Evans' form} \]

\[ \text{multi.-roots} \quad b \frac{da}{ds} - a \frac{db}{ds} = 0 \]

\[ (s+10)(s+1) - s(s+1) = 0 \]

\[ s^2 + 20s + 10 = 0 \]

\[ s = -10 \pm 3\sqrt{10} = [-0.51, -19.49] \]

\[ R = 3\sqrt{10} = 9.49 \]

\[ -10 + 9.49j \quad \text{p+B} \]

\[ -10 - 9.49j \quad \text{p+B'®} \]
B. graphical construction

\[ K_p = \frac{l_1 \cdot l_2}{l_3} \]

\[ l_1 = 0.51 \quad l_2 = 0.49 \quad l_3 = 9.49 \]

\[ K_p = \frac{0.51 \times 0.49}{9.49} = 0.026 \]

\[ l_1 = \sqrt{10^2 + (9.49)^2} = 13.79 \]
\[ l_2 = \sqrt{9^2 + (9.49)^2} = 13.08 \]
\[ l_3 = 9.49 \]

\[ K_p = \frac{13.79 \times 13.08}{9.49} = 19.00 \]

\[ l_1 = 19.49 \quad l_2 = 18.49 \]
\[ l_3 = 9.49 \]

\[ K_p = \frac{19.49 \times 18.49}{9.49} = 37.95 \]
C. \[ t_r = \frac{1.8}{\omega_n} \]

\[ \omega_n = 0.51 \text{ s}^{-1} \quad t_r = \frac{1.8}{0.51} = 3.53 \text{ s} \]

\[ \omega_n = \sqrt{10^2 + (9.49)^2} = 13.8 \text{ s}^{-1} \]

\[ t_r = \frac{1.8}{13.8} = 0.130 \text{ s} \]

\[ \omega_n = 19.49 \text{ s}^{-1} \]

\[ t_r = \frac{1.8}{19.49} = 0.092 \text{ s} \]

alternative soln.: Use \( \omega_n = \sqrt{10k_p} \)

and \( k_p \) from part B to find \( \omega_n \).

D. with closed-loop poles at \( B \) and \( B' \)

\[ B -10 \]

\[ \text{9.49j} \]

\[ \text{15}\]

\[ \omega_n = 13.8 \]

\[ \sin \theta = \frac{8}{13.8} = 0.72 \]

\[ \sqrt{1 - \frac{s^2}{13.8^2}} = 0.69 \]

\[ M_p = e^{-\frac{10 \times 0.72}{13.8}} = e^{-\frac{\pi(0.72)}{0.69}} = 0.038 \]

(3.8% overshoot)

E. The equations for \( t_r \) and \( M_p \) assume dominant poles and neglect the influence of other poles and zeros. Because of the closed-loop zero at \( s = -10 \), the rise time, \( t_r \), will be shorter, but the overshoot, \( M_p \), will be greater than predicted.
3. Frequency response

A. \[ D(s) G(s) = \frac{k_p (s+10)}{s(s+1)} = \frac{10k_p}{s} \frac{1+s/10}{1+s} \]

Bode form

\[ D(j\omega) G(j\omega) = \frac{10k_p}{j\omega} \frac{(1+j\omega/10)}{(1+j\omega)} \]

\[ M(\omega) = \frac{10k_p}{\omega} \frac{\sqrt{1+(\omega/10)^2}}{\sqrt{1+\omega^2}} \]

\[ \phi(\omega) = -90^\circ - \tan^{-1} \omega + \tan^{-1}(\omega/10) \]

B. See next page.

C. \( M = 1 \) at \( \omega = \omega_c \)

\[ \frac{10}{\omega_c} \times \frac{\sqrt{1+(\omega_c/10)^2}}{\sqrt{1+\omega_c^2}} = 1 \]

\[ \frac{100}{\omega_c^2} = \frac{1 + \omega_c^2/100}{1 + \omega_c^2} = 1 \]

\[ 100 + \omega_c^2 = \omega_c^2 (1 + \omega_c^2) \]

\[ \omega_c^4 = 100 \]

\[ \omega_c = \sqrt{10} = \sqrt[4]{10} = 3.16 \text{ rad/s} \]

\[ \phi(\sqrt{10}) = -90^\circ - \tan^{-1} \sqrt{10} + \tan^{-1}(1/\sqrt{10}) \]

\[ = -90^\circ - 72.46^\circ + 17.55^\circ = -144.9^\circ \]

\[ PM = 180^\circ + \phi = 35.1^\circ \]

\[ \hat{g} = \frac{PM}{100} = \frac{35.1}{100} \]

From problem 1, \( g = \frac{1+k_p}{2\sqrt{10k_p}} = \frac{1}{100} = 0.32 \).
Bode plot

\[ D(j\omega)G(j\omega) = \frac{10}{j\omega} \frac{1 + \frac{j\omega}{10}}{(1 + j\omega)} \]