In the case of a single-input single-output (SISO) LTI system, the relation between the input and output in the s-domain can be represented by a rational function called a transfer function.

Example
Spring-mass-damper system

\[ G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \]
TRANSFER FUNCTIONS

Example
Spring-mass-damper system

\[ m\ddot{x} + c\dot{x} + kx = F \]

Input: F  
Output: x

\[ G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \]

FREQUENCY RESPONSE

Assume the transfer function \( G(s) \) is asymptotically stable

\[ x(t) = \sin(\omega t) \quad y(t) = A\sin(\omega t + \phi) \]

Answer
\[
A = |G(j\omega)| \\
\phi = \angle G(j\omega)
\]
COMPLEX NUMBERS

Every complex number has a magnitude and phase:

\[ |z| = \sqrt{a^2 + b^2} \]
\[ \angle z = \tan^{-1}\left(\frac{b}{a}\right) \]

\[ z = a + bj \]

FREQUENCY RESPONSE

\[ G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \]

Input: \( F = \sin(\omega t) \)

Output:

\[ x = |G(j\omega)| \sin(\omega t + \angle G(j\omega)) \]

\[ |G(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \]
\[ \angle G(j\omega) = -\tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \]
UNDERSTANDING BODE PLOTS

Bode plot of spring-mass-damper system:
Plot $|G(j\omega)|$ and $\angle G(j\omega)$ as a function of $\omega$
BODE PLOTS

The magnitude plot of \( G(s) = \frac{1}{s} \) has a slope of -20 dB/dec.

\[ |G(j\omega)| = \frac{1}{\omega} \]
\[ \angle G(j\omega) = -\tan^{-1}(\omega / 0) = -90^\circ \]

\[ |G(j\omega)| = \omega \]
\[ \angle G(j\omega) = 90^\circ \]

When \( \omega \) changes by a factor of 10, \( |G(j\omega)| \) changes by a factor of 10.

\[ |G(j\omega)| = \frac{1}{j\omega} = \frac{1}{\omega} \]
\[ |G(j1)| = 1 \]
\[ |G(j10)| = \frac{1}{10} = 0.1 \]

\[ 20\log |G(j\omega_2)| - 20\log |G(j\omega_1)| \]
\[ = 20\log \left| \frac{G(j\omega_2)}{G(j\omega_1)} \right| = 20\log \frac{1}{10} = -20 \]
BODE PLOTS

Relative degree of transfer function
= order of denominator - order of numerator

\[ G(s) = \frac{1}{s} \]
Relative degree = 1
Transfer function rolls off at -20 dB/dec

\[ G(s) = \frac{1}{Ts + 1} \]
Relative degree = 1
Transfer function rolls off at -20 dB/dec

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
Relative degree = 2
Transfer function rolls off at -40 dB/dec

\[ G(s) = \frac{1}{Ts + 1} \]

\[ |G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2 + 1}} \]

\[ \angle G(j\omega) = \tan^{-1}(-T\omega) \]
BODE PLOTS

1. \( G(s) = Ts + 1 \)

\[
|G(j\omega)| = \sqrt{T^2 \omega^2 + 1}
\]

\[
\angle G(j\omega) = \tan^{-1}(T\omega)
\]

2. \( G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \)

\[
|G(j\omega)| = \frac{\omega_n^2}{\sqrt{\omega_n^4 - \omega^4} + (2\zeta\omega_n\omega)^2}
\]

\[
\angle G(j\omega) = \tan^{-1}\left(-\frac{2\zeta\omega\omega_n^2}{\omega_n^2 - \omega^2}\right)
\]
IMPORTANCE OF BODE PLOTS

- Experimentally determining the dynamic model for a system
- Design of vibration isolation mounts
- Design of sensors
- Design of actuators
- Signal processing filters
- Control system design
- Countless other applications …….

VIBRATION ISOLATION

Designing vibration isolation mounts for a machine

F - mean 5000 N and sinusoidal 1000 N at 25Hz

Machine

Structure

Objective

- Motion of the machine should be less than ±1 mm from equilibrium
- At 25 Hz, less than 250 N of force should be transmitted to the structure
**VIBRATION ISOLATION**

Designing vibration isolation mounts for a machine

\[ F_t = c\ddot{x} + kx \]

**Motion of machine**

\[ G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \]

**Force transmitted to structure**

\[ H(s) = \frac{F_s(s)}{F(s)} = \frac{cs + k}{ms^2 + cs + k} \]

\[ m\ddot{x} + c\dot{x} + kx = F \]

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**VIBRATION ISOLATION**

Motion of machine

\[ G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{k} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ m\ddot{x} + c\dot{x} + kx = F \]
VIBRATION ISOLATION

Force Transmitted

\[ H(s) = \frac{F_i(s)}{F(s)} = \frac{cs + k}{ms^2 + cs + k} = \frac{1}{k} \frac{\omega_n^2 \left( \frac{c}{k} s + 1 \right)}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

Conclusions

- A very hard spring can restrain motion to be less than ±1 mm
- But a hard spring increases the high frequency forces transmitted to the structure
IMPORTANCE OF BODE PLOTS

- Experimentally determining the dynamic model for a system
- Design of vibration isolation mounts
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- Countless other applications .......

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