Transform Solutions to LTI Systems – Part 4

April 2, 2013

**Final Value Theorem**

Given \( F(s) \), how can we find \( \lim_{t \to \infty} f(t) \)

Final Value Theorem (FVT):

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]

When is the FVT applicable?

1). \( F(s) \) should have no poles in the right half of the complex plane (Real part should not be +v).

2). \( F(s) \) should have no poles on the imaginary axis, except at most one pole at \( s=0 \).

**Examples:**

a) \( F(s) = \frac{A}{s} \), find \( \lim_{t \to \infty} f(t) \)

\[
\lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{A}{s} = A
\]

b) \( F(s) = \frac{A}{s^2} \), FVT cannot be applied.

Note: \( \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} = At \).

Final value is not defined.
c) \( F(s) = \frac{\omega}{s^2 + \omega^2} \)

\[
\lim_{s \to 0} F(s) = \lim_{s \to 0} \frac{s \cdot \omega}{s^2 + \omega^2} = 0
\]

\( \mathcal{L}^{-1}\{F(s)\} = \sin \omega t \)

The poles are \( s^2 + \omega^2 = 0 \), \( s^2 = -\omega^2 \), \( s = \pm j\omega \)

Real part=0

Hence the FVT cannot be applied.

d) \( F(s) = \frac{s}{s^2 + \omega^2} \)

The FVT cannot be applied.

**Example on use of IVT and FVT**

\( F(s) = \frac{s^2 + 2s + 4}{s^3 + 3s^2 + 2s} \)

IVT: Can it be applied? YES

\[
f(0^+) = \lim_{s \to \infty} s F(s) = \lim_{s \to \infty} \frac{s^3 + 2s^2 + 4s}{s^3 + 3s^2 + 2s} = \lim_{s \to \infty} \frac{1 + \frac{2}{s} + \frac{4}{s^2}}{1 + \frac{3}{s} + \frac{2}{s^2}} = 1
\]

FVT: Can it be applied?

Poles are obtained from:
\[
s^3 + 3s^2 + 2s = 0
\]

or 
\[
s(s^2 + 3s + 2) = 0
\]

or 
\[
s(s + 2)(s + 1) = 0
\]

Note: A second order polynomial with positive coefficients always has roots with \(-ve\) real parts.

e.g. \(ms^2 + bs + k = 0\) \(\Rightarrow\) the poles are always stable.

A higher order polynomial (3\(^{rd}\) order or higher) need not be stable if all coefficients are positive.

However, even if one coefficient is \(-ve\), the system will be unstable.

(will have at least one pole with +ve real part)

The FVT is applicable in this example.

\[
limit_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s^3 + 2s^2 + 4s}{s^3 + 3s^2 + 2s} = \lim_{s \to 0} \frac{s^2 + 2s + 4}{s^2 + 3s + 2} = \frac{4}{2} = 2
\]

Example:

\[
F(s) = \frac{5s^2 + 8s - 5}{s^2(s^2 + 2s + 5)}
\]

Find \(f(t)\), feature: Repeated pole at \(s=0\).

\[
F(s) = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 2s + 5}
\]

After calculation, it turns out

\[
A = -1, B = 2, C = -2, D = 2
\]

Hence
\[
F(s) = \frac{-s + 2}{s^2} - \frac{2s - 2}{s^2 + 2s + 5} = -\frac{1}{s} + \frac{2}{s^2} - \frac{2(s - 1)}{(s + 1)^2 + 4}
\]

\[
= -\frac{1}{s} + \frac{2}{s^2} - \frac{2(s + 1 - 2)}{(s + 1)^2 + 4}
\]

\[
= -\frac{1}{s} + \frac{2}{s^2} - \frac{2(s + 1)}{(s + 1)^2 + 2^2} + \frac{4}{(s + 1)^2 + 2^2}
\]

\[
= -\frac{1}{s} + \frac{2}{s^2} - \frac{2(s + 1)}{(s + 1)^2 + 2^2} + \frac{2}{(s + 1)^2 + 2^2}
\]

\[
f(t) = -1 + 2t - 2e^{-t}\cos 2t + 2e^{-t}\sin 2t
\]

**Example:** To illustrate how to handle numerator and denominator of the same order.

\[
\frac{2s^2 + 7s + 8}{s^2 + 3s + 2} = \frac{2(s^2 + 3s + 2) + s + 4}{s^2 + 3s + 2}
\]

\[
= \frac{2(s^2 + 3s + 2)}{s^2 + 3s + 2} + \frac{s + 4}{s^2 + 3s + 2} = 2 + \frac{s + 4}{(s + 2)(s + 1)}
\]

\[
= 2 + \frac{A}{s + 2} + \frac{B}{s + 1} = 2 - \frac{2}{s + 2} + \frac{3}{s + 1}
\]

Hence: \(f(t) = 2\delta(t) - 2e^{-2t} + 3e^{-t}\)
Standard Representation of a Second Order System

\[ m \ddot{x} + b \dot{x} + kx = F(t) \]

In the Laplace domain,

\[ m \left( s^2 X(s) - sx(0) - \dot{x}(0) \right) + b \{ sX(s) - x(0) \} + kX(s) = F(s) \]

\[ (ms^2 + bs + k)X(s) = F(s) + (ms + b)x(0) = m\dot{x}(0) \]

\[ X(s) = \frac{1}{ms^2 + bs + k} F(s) + \frac{(ms + b)x_0 + mv_0}{ms^2 + bs + k} \]

\[ \frac{1}{ms^2 + bs + k} \rightarrow \text{Zero} - \text{state response} \]

\[ \frac{(ms + b)x_0 + mv_0}{ms^2 + bs + k} \rightarrow \text{Zero} - \text{input response} \]

For a stable system, the zero input response \( \rightarrow 0 \) as \( t \rightarrow \infty \).

The zero-state response need not converge to zero as \( t \rightarrow \infty \).

It will have some terms that converge to zero and some terms that do not converge to zero.

For example: if \( F(s) = \frac{F_0}{s} \)

\[ \lim_{t \to \infty} f(t) = \frac{F_0}{k} \ ( \text{does not converge to zero}) \]

The terms that do not converge to zero constitute the **steady state response** of the system, and all the terms that converge to zero constitute
the transient response.

If the force is a sinusoid, the steady state response will be a sinusoid.
If the force is some others bounded periodic function, the steady state response will be a bounded periodic function.

The transfer function \( \frac{1}{ms^2 + bs + k} \) can be written in the following standard 2nd order form:

\[
\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

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Continuing with the previous m, k, b system.
If the initial conditions are zero,

\[
X(s) = \frac{1}{ms^2 + bs + k} F(s)
\]

To write in the standard second order form,

1). Divide numerator and denominator by m,

\[
X(s) = \frac{\frac{1}{m}}{\frac{s^2}{m} + \frac{b}{m}s + \frac{k}{m}} F(s)
\]

Coefficient of \( s^2 \) in the denominator is now 1.

2). Need to make the constant terms in the numerator and denominator equal
\[ X(s) = \frac{1}{k} \frac{k}{s^2 + \frac{b}{m} s + \frac{k}{m}} F(s) \]

Compare with

\[ \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

We get

\[ \frac{k}{m} = \omega_n^2, 2\zeta \omega_n = \frac{b}{m} \]

\( \omega_n \): Undamped natural frequency of the system

\[ \zeta = \frac{1}{2\omega_n m} = \frac{1}{\frac{\omega_n^2}{2\sqrt{k/m}}} = \frac{b}{2\sqrt{km}} \]

\( \zeta \) is called the damping ratio.

If \( \zeta > 1 \), the system is said to be over-damped. It has no overshoot and no oscillations.

If \( \zeta < 1 \), the system is said to be under-damped. It has oscillatory behavior and it has overshoot.

To see why, note that the characteristic equation is:

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 \zeta^2 - \zeta^2 \omega_n^2 + \omega_n^2 = 0 \]

\[ (s + \zeta \omega_n)^2 - \zeta^2 \omega_n^2 + \omega_n^2 = 0 \]

\[ (s + \zeta \omega_n)^2 = \omega_n^2 (\zeta^2 - 1) \] \( \cdots \cdots \cdots \cdots \)

If \( \zeta < 1 \), then:

\[ (s + \zeta \omega_n) = \pm j \omega_n \sqrt{1 - \zeta^2}, \quad j = \sqrt{-1} \]
Hence the poles are:

\[ s = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \]

Since

\[ X(s) = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} F(s) \]

If \( F(s) = \frac{1}{s} \) (the response to a step input), then

\[ X(s) = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \cdot \frac{1}{s} \]

After partial fraction expansion, the response will be of the type:

\[ x(t) = x_{ss} + A e^{-\zeta \omega_n t} \sin \omega_d t + B e^{-\zeta \omega_n t} \cos \omega_d t \]

(Likely \( x_{ss} = \frac{F}{k} \) from previous experience)

where \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) \hspace{1cm} (Imaginary part of the poles)

\( \omega_d \) is called the damped natural frequency. Thus the response of the system is oscillatory.

On the other hand, consider \( \zeta > 1 \),

The character eqn gives:

\[ (s + \zeta \omega_n)^2 = \omega_n^2 (\zeta^2 - 1) \]

Hence the poles are:

\[ s = -\zeta \omega_n \pm \sqrt{\omega_n^2 (\zeta^2 - 1)} \]

\[ s = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

Hence the response to

\[ X(s) = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \cdot \frac{1}{s} \]
is given by:

\[ x(t) = x_{ss} + Ae^{-(\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1})t} + Be^{-(\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1})t} \]

Thus the response is not oscillatory. No overshoot.

What about \( \zeta = 1 \)?

The value of \( \zeta = 1 \) is called critical damping. It is the transition point between no oscillations and oscillations.

The poles are given in this case by:

\[ s = -\zeta \omega_n = -\omega_n \]

Hence

\[ X(s) = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \cdot \frac{1}{s} = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \cdot \frac{1}{s}(\text{for } \zeta = 1) = \frac{1}{k} \cdot \frac{\omega_n^2}{(s + \omega_n)^2} \cdot \frac{1}{s} \]

After partial fraction expansion and inverse Laplace transforms, the response is found to be:

\[ x(t) = x_{ss} + Ae^{-\omega_n t} + Bte^{-\omega_n t} \]

(Why? \( \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at} \))

Again the response is exponential \( \Rightarrow \) no oscillations.
Frequency Response of Linear Time Invariant Systems

Complex Numbers: Recall that every complex number has a magnitude and a phase.

Example: \( z = a + bj \), \( j = \sqrt{-1} \)

\( a \) is called the real part of \( z \), \( a = \text{Re}(z) \)

\( b \) is called the imaginary part of \( z \), \( b = \text{Im}(z) \)

Magnitude of \( z \): \( |z| = \sqrt{a^2 + b^2} = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} \)

Phase of \( z \): \( \angle z = \tan^{-1} \left( \frac{b}{a} \right) = \tan^{-1} \left( \frac{\text{Im}(z)}{\text{Re}(z)} \right) \)

Both the magnitude and phase of a complex number are real.

What is the steady state response of any LTI system for a sinusoidal input of frequency \( \omega \)?

Assume that the system is stable: All its poles have negative real parts.

For example:
\[ X(s) = \frac{1}{ms^2 + bs + k} F(s) \]

If \( F(s) = \frac{\omega}{s^2 + \omega^2} \) (sinusoid)

Then \[ X(s) = \frac{1}{ms^2 + bs + k} \cdot \frac{\omega}{s^2 + \omega^2} \]

After partial fraction expansion, and inverse Laplace transforms, we will find:

\[ x(t) = A e^{-\zeta \omega_n t} \sin \omega_d t + B e^{-\zeta \omega_n t} \cos \omega_d t + C \sin \omega t \]