Castigliano's theorem can also be used to solve for deflections in structures which are statically indeterminate.

Example:

A fixed-fixed beam is subjected to a uniformly distributed load of \( w \):

Find the deflection at \( x = \frac{l}{4} \).

Solution:

Since no point load is applied at \( \frac{l}{4} \), we add a fictitious force, \( Q \), at that point.
When we go to find the reactions at the wall, we realize that the problem is statically indeterminate.

FBD (replacing the distributed load with its resultant for the static analysis):

(Note that we can't use symmetry arguments because Q is not centered, even though it is a fictitious force)

From statics:

\[ \Sigma M = 0: \quad M_L + M_R - \frac{1}{2} Q - \frac{wL^2}{2} + R_R L = 0 \]

\[ \Sigma F_y = 0: \quad R_L - Q - wL + R_R = 0 \]

2 equations, 4 unknowns \((R_L, R_R, M_L, M_R)\) \(\Rightarrow\) Statically indeterminate by order 2!

We can now choose any 2 of the 4 unknown generalized forces and assume them to be known external loads.

We can solve for the 2 statically indeterminate forces using Cattarla's theorem by applying the condition that the deflection where they are applied is 0.

Note: Whatever 2 forces are chosen to be the statically indeterminate forces, those 2 are assumed to be independent of Q!
We will choose \( M_L \) \& \( P_L \) to be the statically indeterminate forces (although we could have chosen any 2 from the set \{ \( P_L \), \( P_R \), \( M_L \), \( M_R \) \}).

Solve for the remaining 2 unknowns using the static equations:

\[
R_R = Q + wL - R_L
\]

\[
M_R = -M_L + \frac{Q}{L} + \frac{wL^2}{2} - R_L L
\]

Determine internal loads using method of sections:

\[0 < \alpha < \frac{L}{4}\] 

\[
M_{b1} = -M_L + R_L \alpha - \frac{1}{2} w \alpha^2
\]

\[\frac{L}{4} < \alpha < L\]

\[
M_{b2} = -M_L + R_L \alpha - \frac{1}{2} w \alpha^2 - Q(\alpha - \frac{L}{4})
\]

Then:

\[
U = \int_0^{\frac{L}{4}} \frac{M_{b1}^2}{2EI} \, d\alpha + \int_{\frac{L}{4}}^{L} \frac{M_{b2}^2}{2EI} \, d\alpha
\]
Now, we can obtain an equation relating $R_L$ to $M_L$ by constraining the deflection at the support to $\phi$ using Castigliano's theorem:

$$\delta_1 = \frac{3U}{3R_L} = \phi$$

$$\phi = \frac{3U}{3R_L} = \frac{1}{\beta_1} \left( \int_0^l M_{b1} \frac{3M_{b1}}{3R_L} \, dx + \int_0^l M_{b2} \frac{3M_{b2}}{3R_L} \, dx \right)$$

Note: Computations are minimized by taking the partial derivatives before evaluating the integrals.

$$\frac{3M_{b1}}{3R_L} = \alpha$$

$$\frac{3M_{b2}}{3R_L} = \alpha$$

Furthermore, we can disregard fictitious forces $Q$ while evaluating the statically indeterminate reactions.

$$\phi = \int_0^l \left( -M_L + R_L \alpha - \frac{1}{2} w x^2 \right) \alpha \, dx + \int_0^l \left( -M_L + R_L \alpha - \frac{1}{2} w x^2 \right) \alpha \, dx$$

$$= \int_0^l \left( -M_L \alpha + R_L \alpha^2 - \frac{1}{2} w x^2 \alpha \right) \, dx$$

$$= \left. \left( -\frac{1}{2} M_L \alpha^2 + \frac{1}{2} R_L \alpha^3 - \frac{1}{8} w x^4 \right) \right|_0^l$$

$$= -\frac{1}{2} M_L \alpha^2 + \frac{1}{2} R_L \alpha^3 - \frac{1}{8} w l^4$$

or:

$$\phi = -\frac{1}{2} M_L + \frac{1}{2} R_L - \frac{1}{8} w l^2 = \phi \quad (\dagger)$$
A second relation between $R_L$ & $M_L$ can be obtained by constraining the rotation at the left support to be $\phi$:

$$\phi = \frac{\partial U}{\partial M_L} = \phi$$

$$\phi = \frac{\partial U}{\partial M_L} = \frac{1}{l} \left( \int_0^l M_{b1} \frac{\partial M_{b1}}{\partial M_L} \, dx + \int_0^l M_{b2} \frac{\partial M_{b2}}{\partial M_L} \, dx \right)$$

$$\frac{\partial M_{b1}}{\partial M_L} = -1$$

$$\frac{\partial M_{b2}}{\partial M_L} = -1$$

Again, disregard $\phi$ while solving for statically indeterminate reactions:

$$\phi = \int_0^l (-M_L + R_L x - \frac{1}{2} w x^2) \, dx + \int_0^l (-M_L + R_L x - \frac{1}{2} w x^2) \, dx$$

$$= \int_0^l (M_L - R_L x + \frac{1}{2} w x^2) \, dx = (M_L x - \frac{1}{2} R_L x^2 + \frac{1}{6} w x^3) \bigg|_0^l$$

$$= M_L - \frac{1}{2} R_L l^2 + \frac{1}{6} w l^3$$

or:

$$M_L - \frac{1}{2} R_L l + \frac{1}{6} w l^2 = 0 \quad (2)$$

Determine $R_L$ & $M_L$ by solving (2) & (2) simultaneously:

$$2 \times (2): \quad -M_L + \frac{3}{2} R_L l - \frac{1}{2} w l^2 = 0$$

$$(2): \quad + (M_L - \frac{1}{2} R_L l + \frac{1}{6} w l^2 = 0)$$

$$\frac{1}{2} R_L l - \frac{1}{2} w l^2 = 0$$

$$R_L = \frac{1}{2} w l \quad (\text{validated by symmetry if } Q \text{ is disregarded})$$

From (2):

$$M_L = \frac{1}{2} R_L l - \frac{1}{2} w l^2 = \frac{1}{2} (\frac{1}{2} w l) l = \frac{1}{4} w l^2$$

$$M_L = \frac{1}{12} w l^2 \quad (\text{agrees with J&M Appendix D-3 #3, after adjusting for J&M's sign convention})$$
Now that we know \( R_2 \) & \( M_r \), we can sub them into our bending moment expressions:

\[
M_{b1} = -\frac{1}{2} wL^2 + \frac{1}{2} wLx - \frac{1}{2} wnx^2
\]

\[
M_{b2} = -\frac{1}{2} wL^2 + \frac{1}{2} wLx - \frac{1}{2} wnx^2 - Q(x - \frac{b}{4})
\]

Now, we can apply Castigliano's theorem a second time to determine the deflection at \( x = \frac{b}{4} \):

\[
61_{x=\frac{b}{4}} = \frac{\partial^2 U}{\partial Q} \bigg|_{Q=0} = \frac{d}{dx} \left( \int_0^{\frac{b}{4}} M_{b1} \, dx + \frac{1}{E} \int_0^{\frac{b}{4}} M_{b2} \, dx \right)
\]

But \( \frac{\partial M_{b1}}{\partial Q} = 0 \); \( \frac{\partial M_{b2}}{\partial Q} = -(x - \frac{b}{4}) \)

Once we have the partial derivatives, we can set \( Q = 0 \):

\[
M_{b2} = -\frac{1}{2} wL^2 + \frac{1}{2} wLx - \frac{1}{2} wnx^2
\]

\[
\begin{align*}
61_{x=\frac{b}{4}} &= \frac{d}{dx} \left( -\frac{1}{2} wL^2 + \frac{1}{2} wLx - \frac{1}{2} wnx^2 \right) \left( \frac{b}{4} - x \right) dx \\
&= \frac{d}{dx} \left( -\frac{1}{16} wL^3 + \frac{5}{24} wL^2x + \frac{1}{12} wLx^2 + \frac{1}{2} wnx^3 \right) dx \\
&= \frac{1}{16} \left( -\frac{1}{48} wL^3 + \frac{5}{24} wL^2x - \frac{1}{12} wLx^2 + \frac{1}{2} wnx^3 \right) dx \\
&= \frac{1}{16} \left( -\frac{1}{48} wL^3 + \frac{5}{24} wL^2x - \frac{1}{12} wLx^2 + \frac{1}{2} wnx^3 \right) \left( \frac{b}{4} - x \right) dx \\
&= \frac{wL^4}{614 + EI} \left( -128 + 640 - 1280 + 768 + 32 - 40 + 20 - 3 \right)
\end{align*}
\]

\[
61_{x=\frac{b}{4}} = \frac{3}{2048} \frac{wL^4}{EI}
\]

 cred: From J&M Appendix D-3#3:

\[
6 = \frac{wx^2}{144 EI} (l-x)^2
\]

When \( x = \frac{b}{4} \), \( 6 = \frac{3}{2048} \frac{L^4}{EI} \)

While this problem could have been solved using the beam tables, this method will also work for cases which are not available from the beam tables. One of the homework problems will demonstrate that Castigliano's method can be used to derive the beam table equations!