TOPOLOGICAL SYNTHESIS AND INTEGRATED KINEMATIC-STRUCTURAL DIMENSIONAL OPTIMIZATION OF A TEN-BAR LINKAGE FOR A HYDRAULIC RESCUE SPREADER

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ABSTRACT

Hydraulic rescue spreaders are used by emergency response personnel to extricate occupants from a vehicle crash. A lighter and more portable rescue spreader is required for better usability and to enable utilization in a variety of scenarios. To meet this requirement, topological synthesis, dimensional synthesis, and an optimization were used to develop a solution linkage. The topological synthesis technique demonstrates that ten links are the minimum possible number that achieves the desired motion without depending primarily on rotation of the spreader jaws. A novel integrated kinematic-structural dimensional synthesis technique is presented and used in a grid-search optimizing the linkage dimensions to minimize linkage mass. The resulting ten-bar linkage meets or exceeds the kinematic performance parameters while simultaneously achieving a near-optimum predicted mass.

INTRODUCTION

Motivation

Hydraulic rescue spreaders, also known by the brand name “jaws of life”, are hydraulically actuated mechanisms used in emergency situations to remove victims trapped inside wreckage, often as a result of automobile accidents. They are composed of a linkage that converts the motion of a hydraulic cylinder to the spreading action of a pair of jaws. A typical rescue spreader and motion schematic is shown in Figure 1. Generally the jaws must exert a large amount of force to deform various metal structures, resulting in large loads throughout the linkage. As the rescue spreader is an emergency tool that must be used with relative speed and ease by a single operator, care must be taken to design the mechanism efficiently to minimize the mass.

FIGURE 1: TYPICAL RESCUE SPREADER [2]

Most existing rescue spreaders utilize a six-bar linkage, and can weigh up to 25 kg depending on the required spreading force and spreading distance. Table 1 lists spreading force and spreading distance for a variety of models from leading manufacturers, and Figure 2 plots the same data to highlight the dependence of these metrics on spreader mass. The majority of existing designs use a hydraulic power supply that is kept stationary and connected to the spreader by long hoses in order to avoid the necessity of the operator carrying extra weight, as would be the case if the power supply were integrated into the

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1 Forces quoted are Lowest Spreading Force (LSF) values as defined in NFPA-1936, as these are the most relevant for typical use scenarios [1].
spreader directly. This creates a constraint on the use of the spreader as the operator is now tethered to a semi-fixed point. If the power supply were integrated into the spreader without exceeding acceptable weight limits, the operator would enjoy greater freedom of action. Furthermore, if the weight were substantially reduced, additional usage scenarios would become feasible, such as rapid deployment by air to remote areas, an application of interest to the military.

### Problem Statement

The goal of the work presented in this paper is to find the lightest possible mechanism that will withstand the necessary forces with an appropriate safety factor while also meeting certain kinematic performance requirements. These requirements are:

1. The variation in mechanical advantage between the hydraulic actuator and the jaws over the course of the stroke must be small in order to avoid having to design for a large maximum force at the jaw tips while having to accept a small minimum force. Variation is defined as the maximum deviation from the mean divided by the mean, as illustrated in Figure 3. Variation is capped at 15%.

2. The angular deviation of the jaw tips from a perfectly linear outward spreading motion must be small, so that the entire tool is not pushed forward or backward undesirably by the action of the jaws. Deviation is capped at $20^\circ$.

3. The amount of rotation of the jaws over the course of the stroke must be small, as if the face of the jaw gripping the target material becomes too inclined slippage can occur. Rotation is capped at $40^\circ$.

4. No transmission angle is permitted to be less than $30^\circ$ at any point in the motion.

Note that these are (somewhat arbitrary) constraints and these quantities are not required to be minimized. Only the mass must be minimized as a single-objective optimization with constraints is more tractable than a multi-objective optimization.
Existing rescue spreader designs are all slight variations on the same basis: a bilaterally symmetric six-bar mechanism actuated by a single hydraulic cylinder. Such a design introduces undesirable structural consequences, as will be seen shortly. The synthesis procedure presented here attempts to start with as many free design choices as possible so as to maximize the chance of finding the lightest possible acceptable design, and thus includes possibilities that differ significantly from the standard design. To this end, the initial constraints on the problem are limited to the following set:

**Kinematic:**
1. The linkage is composed of a number of rigid bodies, or links.
2. Exactly one link is grounded, i.e. required to neither rotate nor translate with respect to the reference frame.
3. The linkage is planar with 1 degree of freedom (DOF).
4. The linkage is driven by a single linear actuator.
5. Links are connected to each other solely by revolute joints, with the exception of a single prismatic joint representing the linear actuator.
6. Exactly two of the links are designated as the jaws, which must both contain a point that is unobstructed by the rest of the linkage, designated as the tips.
7. The jaw tips must move smoothly from a coincident position (“closed position”) to a given distance apart (“open position”) as the position of the actuator changes.

**Mechanical:**
- The linear actuator is hydraulic.
- Subject to a load of 80 kN at each jaw tip, the linkage must not undergo failure, defined as plastic deformation, at any point.
- The jaw tips must start in contact with each other and be 45.72 cm apart in the open position.

Additionally, any candidate solution must also satisfy the four kinematic requirements given previously.

**METHODS**

The process of synthesizing the mass-optimized mechanism can be broken into three parts. First, a topological synthesis was performed, which determined in order of increasing specificity:
- The number of links in the mechanism
- The isomer, given the link number
- The inversion (choice of ground link) given the isomer
- Which links would be jaw links and which joint would be the prismatic joint, given the inversion

The second step was to take the linkage topology thus determined and optimize the geometry via a coupled dimensional-structural synthesis process to achieve the desired kinematics as well as minimum linkage mass.

The third step was the detailed mechanical design of the final product given the topology and geometry, which was a straightforward and non-novel undertaking and thus is not discussed in this paper.

**Topological Synthesis**

The most basic decision to be made was how many links to include in the mechanism. In general, it is reasonable to assume that minimizing the number of bars will help to minimize the overall mass of the mechanism, and thus we wish to determine the lowest number of bars that is valid given the kinematic requirements. We begin by considering Gruebler’s mobility equation for a planar mechanism:

\[ M = 3(L - 1) - 2J \]  

where M is the number of DOF in the mechanism, L is the number of links, and J is the number of full joints. Note that the linkage is required to have only revolute and prismatic joints, which are both types of full joints, and thus we need not consider half joints. It is easily shown that for \( M = 1 \), \( L \) must be an even number [7].

**Symmetry Requirement**

At this point we make an addition to the list of kinematic requirements:

8. The linkage must be bilaterally symmetric both topologically and geometrically.

It is entirely possible to envision linkages that do not meet this requirement, notably the case where the ground link is also one of the jaws, but as both intuition and tradition point in the direction of a symmetrical design it was decided to investigate these first. Letting the y axis in an x-y coordinate system be the line of symmetry of the mechanism, it is now convenient to note a number of corollaries to the kinematic requirements stated so far:
A. From VIII, any link in the linkage must either have an identical partner on the other side of the y axis or centerline, or be itself located on the centerline. If on the centerline without an identical partner, it can only translate along the centerline (y direction), and cannot rotate or translate in the x direction.

B. As there is only one ground link, it must be on centerline as described in A.

C. By similar logic, the prismatic joint must be on centerline, as well as one of the revolute joints if J-1 is odd (or equivalently if J is even).

Given these conditions, it is evident that a two- or four-bar linkage is not possible.

Six-Bar Solutions

We now turn our attention to six-bars. There are only two six-bar isomers, shown in Figure 4:

![Six-Bar Isomers](image)

**FIGURE 4: SIX-BAR ISOMERS**

Beginning with the Stephenson, it is clear that if any link other than 4 or 5 is grounded symmetry is violated. If 4 or 5 is grounded, the y axis must be chosen to bisect both 4 and 5 and both 4 and 5 must always be parallel to the x axis to preserve symmetry. This leaves the 1-2 joint as the only symmetric choice for the prismatic joint, which must function in such a way that 1 and 2 are also always parallel to the x axis. But if 1,2,4, and 5 are always parallel to the x axis (and have fixed length of course), 3 and 6 are unable to rotate, and the linkage cannot move. Thus we conclude the Stephenson is unusable.

Turning to the Watt, we are forced to ground either 1 or 4 to preserve symmetry. There is an additional initial symmetry between {1,2,6} and {3,4,5}, so we need only consider one of these options, say grounding 1. The 1-4 joint is the only possible choice for the prismatic joint. Choosing either \{2,6\} or \{3,5\} as the jaw pair produces a valid linkage. In fact, existing rescue spreaders are Watt six-bars with one of these two variants (e.g. Figure 1 shows an example where the jaws are \{3,5\}).

Jaw Motion

Having determined that six is the lowest number of bars possible given the additional requirement VIII (symmetry), an important observation can be made regarding the six-bar solutions. Link 1, being grounded, obviously can neither translate in the x direction nor rotate, and from Corollary A neither can Link 4. Thus either choice of jaws, \{2,6\} or \{3,5\}, means the jaws are connected by a revolute joint to a link that can only move in the y direction or not at all, and therefore it is not possible for the entire jaw to translate in the x direction. Consequently, the required x-translation of the jaw tips must be caused solely by rotation of the jaws. But as was stated on the first page, the jaws cannot be permitted to rotate very much. Thus the jaws must be very long in order to produce a large degree of x-translation of the tips with a minimal amount of rotation. This in turn has important structural consequences. A large load is applied transversely to the long jaws, resulting in what is essentially a long cantilever beam in bending. This is a highly unfavorable loading scenario and forces the jaws to be quite massive in order to withstand the large bending moment generated. In fact, the jaws are responsible for a large portion of the total mass of all existing designs. It was hypothesized that if the kinematics could be altered to permit the jaws to translate in the x direction as well as to rotate, thus eliminating the need for excessive length, overall mass might be reduced. We thus wish to explore the effects of adding another kinematic constraint:

IX. The motion of the jaws must include the ability to translate in a direction perpendicular to the axis of symmetry (i.e. in the x direction), although complex motion that also includes rotation and/or y-translation is allowable as well.

which results in two additional corollaries:

D. The jaws cannot share a joint with the ground link, these joints must be revolute given V and VIII, and this would thus confine the jaws to pure rotation in violation of IX. Similarly, if any link is confined to centerline as in A, the jaws cannot share a joint with that link either.

E. In order to satisfy both VIII and IX, it is evident the jaws cannot be connected to each other.
Eight-Bar Solutions

As the new constraint was introduced specifically to disqualify the six-bar solutions with their non-translating jaws, we must next investigate eight-bar linkages. It is now shown that no suitable eight-bar solutions exist. The proof of this claim is accompanied by a series of figures using the symbols shown in Figure 5:

![FIGURE 5: TOPOLOGICAL DIAGRAM KEY](image)

We begin by placing the ground link. It must be on centerline as specified by Corollary B. As there are now seven remaining links to placed, it follows from Corollary A that 1, 3, 5 or 7 additional links must be on centerline and the rest in pairs on either side of the centerline. It is easily shown that placing 3 or more additional links on centerline fails quickly. Thus we now have the ground link and one additional link on centerline, as shown in Figure 6:

![FIGURE 6: GROUND LINK AND SECOND LINK](image)

We now have six more links to place, which must be in three pairs on either side of centerline. One of these pairs must be the jaws. The jaws must each connect to at least two other links in order to form a closed kinematic chain, which is necessary for the mechanism to have a single DOF. However, the jaws cannot connect to links 1 or 2 due to Corollary D, or to each other by E. Also, the jaws cannot connect to links on the opposite side of centerline as they must be able to translate off centerline. Thus each jaw is forced to be a binary link connected to the other two links on the same side of centerline, as shown in Figure 7:

![FIGURE 7: ALL LINKS PLACED, JAWS DETERMINED](image)

Next, C requires that the prismatic joint be on centerline, so clearly it must be placed connecting 1 and 2.

For the next step, we first note that no link can be connected to both 1 and 2, as since 1 and 2 are required to not rotate this would eliminate motion between them. Next, it must be the case that both 1 and 2 are each connected to at least two other links to form a closed kinematic chain. As they are already connected to each other via the centerline prismatic joint, and there is no way to symmetrically have only two joints, both 1 and 2 must be at least ternary links. Given the two requirements just discussed, as well as the requirement from D that the jaws not connect to 1 or 2, the only option is the topology depicted in Figure 8. By inspection, it can be seen that this linkage has three DOF:
Counting the joints used thus far, we find that there are eight revolute joints and one prismatic joint. From Gruebler’s Equation (1), we see that an eight-bar, 1-DOF mechanism must have exactly ten joints. Thus there remains one joint, which must be a revolute joint, to place. As all eight bars are already placed it must therefore connect two existing, currently unconnected bars. However, there is no possible way to place the last joint that does not either prevent motion of one jaw or violate one of the requirements or corollaries. As the steps leading up to the position of Figure 8 were forced, it is thus revealed that the existence of an eight-bar mechanism satisfying all the requirements is a contradiction and thus impossible.

**Generalization to N-Bar Solutions**

It is apparent that proofs of this nature will rapidly become increasingly difficult for higher numbers of links. However, the following instructive if not rigorously defensible observation can be made. Consider Table 2, which shows the number of joints associated with single-DOF mechanisms of various link numbers. Since there is always one prismatic joint, the number of revolute joints is always one less than the total number of joints. Adding two links to get to the next possible linkage requires adding three joints to cancel out the additional degrees of freedom. Thus the number of revolute joints oscillates between odd and even. When it is even, one can place the prismatic joint on centerline and the revolute joints in off-centerline pairs and satisfy symmetry. When it is odd, this is not possible. This suggest that when the number of links is 6, 10, 14, … , valid linkages can be constructed, but not when the number of links is 8, 12, 16… , although of course high numbers of links are not really practical anyway.

**TABLE 2: LINKS AND JOINTS IN 1-DOF MECHANISMS**

<table>
<thead>
<tr>
<th>Links</th>
<th>Joints</th>
<th>Revolute Joints</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>16</td>
<td>15</td>
</tr>
</tbody>
</table>

**Ten-Bar Solution**

This is something of a moot point as the possibility of a ten-bar solution was proved by the simple expedient of finding one that worked, and going to higher numbers of links would almost certainly result in unacceptably high mechanism mass.

**FIGURE 9: TEN-BAR SOLUTION TOPOLOGY**

The ten-bar topology was arrived at by starting with a pair of four-bars sharing the same ground link and adding the remaining links so as to allow the hydraulic actuator to drive both four-bars. The four-bars are circled in Figure 9.

**Dimensional Synthesis**

With the number of links and topology now determined, the next step was to find the linkage dimensions that would correspond to a minimized mechanism mass while meeting the kinematic performance requirements. Utilizing the half symmetry of the mechanism, it was possible to reduce a complete dimensional description to eleven variables, with a
twelfth, \( \theta_2 \), specifying the position of the mechanism, as shown in Figure 10 (right half only shown due to symmetry):

**FIGURE 10: HALF-LINKAGE AND VECTOR DIAGRAM**

At first glance, it appears that this half-linkage is not allowed, as it crosses the centerline. However, for purposes of dimensional synthesis the centerline now represents a geometrical symmetry plane, whereas during topological synthesis it represented a topological symmetry plane, and was a graphical convenience to represent certain properties of the way the different links connect or do not connect to each other. For example, the topologically symmetrical linkages in Figures 8 and 9 are represented as geometrically separated by the centerline for clarity, but one can easily envision introducing geometrical modifications, including crossing the centerline, that do not alter the topology. Similarly, careful comparison of Figures 9 and 10 will reveal that the topology of the right half of Figure 9 is identical to that of Figure 10.

Turning our attention to the mechanism itself, we see that the half-linkage comprises a six-bar. \( \overline{R}_7 \) represents the crossbar at the top of the hydraulic piston and slides up and down the centerline while remaining horizontal. This action rotates the bell crank, driving the four-bar, of which the jaw is the coupler link. Thus the extension of the piston is transformed into rotation and translation of the jaw. \( \overline{R}_7 \) and the ground link \( \overline{R}_1 \) are both half-links after reduction to half-symmetry, and thus adding the left half of the mechanism back in results in ten bars, not twelve. Also note that as the two mechanism halves overlap in-plane, they must in reality operate in two separate but adjacent parallel planes.

**Problem Reduction**

With the action of the mechanism understood, it is apparent that there are thirteen degrees of freedom that must be optimized with respect to: eight lengths and three angles to define the linkage geometry, and the initial and final values of \( \theta_2 \) to determine the range of motion. One degree of freedom can be eliminated by the assumption that the difference between the start and end values of \( \theta_2 \) is fixed at 120 degrees. This is roughly the largest amount of rotation of the bell crank that is allowable based on transmission angle considerations. A smaller rotation is undesirable since it would result in less jaw movement, requiring a scaled up mechanism to meet the target jaw tip travel, making all the links bigger and thus more massive.

The problem is thus reduced to twelve degrees of freedom. A position analysis with a single derivative is required. The position is needed to meet kinematic constraints 2 and 3, which concern the path of the jaw tip. The derivative is needed for constraint 1, which concerns the mechanical advantage variation, as the instantaneous mechanical advantage is proportional to the ratio of velocities of the piston and jaw tip.

**Grid-Search Analysis**

Having defined the degrees of freedom that must be optimized, a brief discussion of traditional kinematic synthesis methods is in order. Erdman and Sandor [8] define three task classifications for kinematic synthesis, known as function, path, and motion generation. The present problem is one of motion generation, where the motion of a line segment on the linkage is constrained such that it assumes some sequential set of specified positions. Specifically, the path of the jaw tip and angle of the jaw must both be controlled. As will be seen in the Linkage Decomposition section, there is also a secondary function generation problem, which must however be solved in combination with the main motion generation problem and thus cannot be solved in isolation using analytical methods. The analytical approach to the motion generation problem will result in an overconstrained, and hence unsolvable, system of equations if more than five precision positions are specified [8]. Five positions are not sufficient to adequately constrain the motion of the jaw, a problem that can be addressed by finding a least-squares solution to a path specified by more than five points, as in the work of Jun et. al. [9]. However, the exact motion of the jaw is not important as long as it is within the acceptability constraints, so it makes more sense to focus on the...
main goal of mass minimization rather than unnecessarily constrain the solution to approximate a particular jaw motion.

Instead of directly synthesizing the solution from the problem definition, an alternative “guess and check” type method is possible in which a large number of potential solutions are generated and then analyzed in the hopes of finding a good match. Specifically, a grid-search method can be employed in which every permutation of discrete values of the linkage variables over a selected range is used to construct a large number of potential or candidate solutions. Each candidate solution in turn is solved for positions at a number of discrete steps through its range of motion, yielding approximate position and velocity profiles for the jaw tip, as illustrated in Figure 11. Approximate instantaneous velocity is found from the distance between adjacent tip locations (grey arrows at top). The position is incremented such that the piston (not shown) moves at a constant rate. The position and velocity data is then used to check for compliance with the four kinematic constraints listed in the Problem Statement section. If any of the four are not satisfied, the kinematics are unacceptable and the candidate solution is rejected immediately.

**FIGURE 11: APPROXIMATE POSITION AND INSTANTANEOUS VELOCITY CALCULATION FROM MULTIPLE POSITIONS OF FOUR-BAR**

In other words, a grid search is performed on a subspace of the theoretically infinite parameter space, selected so as to balance comprehensiveness with computation time. In the limit as the size of the subspace goes to infinity and the interval between grid points goes to zero, every possible set of linkage dimensions is examined.

**Linkage Decomposition**

The obvious drawback to this method is that due to the complexity of the linkage there are a large number of grid dimensions to search, posing computation time problems if any but the coarsest search resolution is desired. The problem is so formidable largely because the effects of changing different input variables cannot be decoupled. For example, if there are ten variables and we want to try ten different values for each one, if each variable could be optimized independently this would only require $10 \times 10 = 100$ candidate linkage evaluations. In contrast, if all ten must be optimized simultaneously, the grid search method requires $10^{10} = 10$ billion evaluations to cover all possible permutations.

Although no single variable could be decoupled from all the others, it is possible to decompose the linkage into the two sub-linkages shown in Figure 12, each of which can be analyzed separately. The primary consists of a four-bar with the jaw as the coupler link. The secondary consists of a dyad where the effect of the piston and cylinder can be represented by confining the bottom of the connecting link to a vertical slot (compare to Figure 10).

**FIGURE 12: LINKAGE DECOMPOSITION**

Let us now suppose for analytical convenience that the bell crank, not the piston, is the driving link. As shown in Figure 12, applying some angular velocity profile (let us say a constant one) to the bell crank will result in some linear velocity profiles being produced at the end of the connecting link in the slot and at the jaw tip. It is evident that changing any linkage dimensions in one sub-linkage will alter the resulting output velocity for that sub-linkage but not the other one, that is, the two are decoupled from each other.

Recall that the purpose of the kinematic analysis is to ensure compliance with the four kinematic constraints on mechanical advantage variation, jaw rotation, and jaw tip deviation. The latter two can be analyzed with the primary sub-linkage alone. Mechanical advantage variation is determined by...
comparing the linear velocity profiles of the jaw tip and end of the connecting link given constant angular velocity of the bell crank. Figure 13 shows one possible set of velocity profiles, as well as the resulting mechanical advantage profile given by dividing the secondary by the primary velocity. Recall that the bell crank is always assumed to rotate 120 degrees.

If the two velocity profiles are constant multiples of each other, then mechanical advantage is a constant and variation is zero. Additionally, if one profile is multiplied by a constant, then the M.A. profile will be multiplied by a constant as well without changing the variation. The important consequence is that M.A. variation is invariant under uniform scaling of one or both sub-linkages. The other two quantities of interest, jaw rotation and jaw tip deviation, are angular quantities and thus are also invariant under uniform scaling. Thus for kinematic purposes the actual size of the sub-linkages, and even the relative size of the two, doesn’t matter. This effectively allows two more variables to be eliminated from the grid search by arbitrarily normalizing both sub-linkages, reducing the problem to ten grid dimensions.

After linkage decomposition and sub-linkage normalization, it becomes convenient to re-define the linkage geometry shown in Figure 10. The new geometry is shown in Figures 14 and 15:

For the primary sub-linkage, the set \( \{r_{1x}, r_{1y}, r_3, r_4, r'_3, \theta_3\} \) fully defines the geometry.\(^2\) The sub-linkage is normalized with respect to \( r_2 \), which is set to length 1 and thus not a variable. \( \theta_2 \) is not needed to specify the geometry but rather determines the position of the linkage along its 1 DOF. Thus an initial value of \( \theta_2 \) is needed to specify a start point, bringing the total to seven input variables, but since bell crank rotation is assumed to always be 120 degrees the final value need not also be specified. \( \theta_3 \) can be eliminated as an independent variable, as will be explained shortly, reducing the input variables to six: \( \{r_{1x}, r_{1y}, r_3, r_4, r'_3, \theta_{2,1}\} \).

For the secondary sub-linkage, the set \( \{r_p, \theta_p, r_c, \theta_c\} \) fully defines the geometry and position. Note that the sub-linkage has been reflected across the symmetry axis in order to define

\(^2\) Throughout this paper, \( \overrightarrow{R} \) refers to a full two-dimensional vector, whereas \( r \) refers to the length of vector \( \overrightarrow{R} \), and \( r_x \) refers to the length of the x component of \( \overrightarrow{R} \).
angles more conveniently during analysis. The variable $r_A$ is left as a dependent quantity to be solved for once the others are specified. The sub-linkage is normalized with respect to $r_B$, which is set to length 1 and thus not a variable. Thus each secondary is defined by specifying its initial position with three variables, $[\theta_B, r_C, \theta_C]$ and then rotating $\theta_B$ through 120 degrees to simulate the motion.

And so we are down to nine grid dimensions from the original thirteen, due to normalizing each sub-linkage, assuming 120 degrees of bell crank rotation, and the elimination of $\theta_3$, which will now be justified. Consider the primary sub-linkage four-bar in its initial position with $[r_{1x}, r_{1y}, r_3, r_4, \theta_{2x}]$ specified (and $r_2$, which is always length 1). It appears that two more variables are needed to locate the position of the jaw tip, but since the tip must be on centerline in the initial position in order to touch its counterpart in the other position of the jaw tip, but since the tip must be on centerline in the initial position in order to touch its counterpart in the other half of the mechanism only one is actually required, eliminating $\theta_3$ in favor of $r'$, which defines the height of the jaw. This vector is required to lie on the centerline in the closed position and connects partway along the $\overrightarrow{R}$ vector as shown on the left side of Figure 16. Having normalized the linkage becomes a bit of a problem at this point. The actual height of the jaw in real units (say cm) can be specified within a small range of 15 cm or so since we know the jaw should be as short as possible (which was the whole rationale for the ten-bar in the first place) as long as it can still function effectively. However, the linkage dimensions are specified in arbitrary units defined such that $r_2$ is length 1, and obviously the units must be consistent.

![FIGURE 16: DETERMINATION OF SCALE FACTOR FROM LINKAGE SPECIFIED IN MIXED UNITS](image)

The dilemma is resolved as follows. The length of $r'$ is specified in cm (the * indicates real units), which corresponds to an unknown number of $r_2$ lengths since the conversion factor is not yet known. The linkage is then solved for the initial and final positions, as shown in Figure 16, which can be done without knowing the height of the jaw. The jaw tip will then have translated in the x direction some distance due to translation of the $\overrightarrow{R}$ vector (labeled T), and some additional spreading distance due to rotation of the jaw through angle $\Phi$ (labeled R).

We know that $T + R = 22.86$ cm, which is half the required spreading distance, and that $R = r' \sin(\Phi)$ cm. Thus T is now known in cm, and also in arbitrary units from solving the normalized linkage, and the conversion factor can now be determined.

With the primary scaling determined, the secondary scaling is found using conservation of energy. The jaw tip must be moved a known distance while exerting a known force, resulting in a known amount of work. Neglecting frictional losses, this must be equal to the work done by the hydraulic actuator. As the operating pressure and cylinder area are specified values and thus known, there is only one piston stroke length that will satisfy this, which constrains the secondary scaling. With the actual linkage dimensions recovered, the next step now becomes possible.

**Kinematic – Structural Coupling**

An important strength of the grid search method was the ability to easily integrate non-kinematic features into the code. Recall that the basic problem is choosing the candidate linkage that minimizes necessary mass. Thus for each grid point or candidate linkage, once the kinematics had been analyzed an additional subroutine was called to attempt to estimate the required mass. A conceptual schematic of the entire procedure is shown in Figure 17, which for visualization purposes uses a 2D grid without linkage decomposition.

The first step was to solve for the forces acting on each link given the linkage geometry and the applied force at the jaw tip. This was done at each of the discrete positions of the linkage used to approximate the motion. The force varied throughout the motion as the geometry changed, so the maximum was selected for each link. At each position, a quasi-static analysis was used, in which it was assumed that the motion of the mechanism was slow enough that a static force balance would be valid.

Once the maximum forces acting on each link throughout the motion of the mechanism were known, each link was examined independently in order to estimate the mass of material required to withstand those forces. Of the six links in the half-mechanism, two are two-force members. A simple uniform axial stress model was used:

$$M = (SF) \rho L \frac{F}{\sigma_y}$$  \hspace{1cm} (2)
where $SF$ is the desired safety factor, $M$ is the link mass, $\rho$ is the material density, $L$ is the length of the link between pin centers, $F$ is the axial load, and $\sigma_y$ is the material yield strength. A buckling analysis was also performed using a hybrid Johnson-Euler model where the appropriate option was selected in each case based on the slenderness ratio of the member [10].

This leaves three three-force members: the jaw, the bell crank, and the piston crossbar ($R_7$ represents half the crossbar), as well as the ground link, which requires special treatment.

The heuristic model for the jaw assumed that the required mass would scale linearly with the lengths of $R_3$ and $R_6$, as well as with the applied force. Several typical geometry and loading scenarios were chosen and analyzed in detail using ANSYS to establish a baseline case to calibrate the model. The part was assumed to be of a uniform thickness given by:

$$t = \frac{F_{\text{max}}}{F_{\text{ref}}} t_{\text{ref}}$$

where $t$ is the thickness, $F_{\text{max}}$ is the maximum of the three pin forces, $F_{\text{ref}}$ is the reference force and $t_{\text{ref}}$ is the reference thickness. The reference quantities were determined by detailed analysis of several test cases in a similar manner to the jaw model. Note that the required safety factor was included in analysis of the reference cases and is thus implicitly present in the final value of $t$. The last step in modeling the bell crank was to take the shape outline thus defined and approximate it by a polygon for computational convenience. The construction of the approximate geometry from kinematic parameters and pin sizing is illustrated in Figure 18:
The heuristic model for the piston crossbar used similar logic as given in Equation 4:

\[ m = m_{\text{ref}} \left( \frac{F}{F_{\text{ref}}} \right) \left( \frac{r}{r_{\text{ref}}} \right)^2 \]  \tag{4}

where \( m \) is the mass of the part, \( m_{\text{ref}} \) is the reference mass, \( F \) is the applied force, \( F_{\text{ref}} \) is the reference force, \( r \) is the length of \( R_r \) and \( r_{\text{ref}} \) is the value of \( r \) in the reference case. The last ratio is squared to account for the effect of loading the crossbar primarily in bending.

The last link to model was the ground link. Looking at the left part of Figure 10, it can be seen that the basic problem is the structural connection of the ground pivots to the hydraulic cylinder. This was assumed to be done separately for the two pivots. A pair of rods extending upwards along the symmetry axis (but offset outwards in the out-of-plane direction from the rest of the mechanism) supports the left pivot and is loaded axially. A beam extends from the side of the cylinder to support the right pivot and is treated in a similar manner to the piston crossbar.

The diameter of the cylinder is needed to calculate the required length of this beam given a known length and angle of \( R \) that locates the pivot center. Given a desired hydraulic operating pressure, and with the necessary force known from solution of the linkage force balance, the internal diameter of the cylinder is easily found. A simple thin-wall hoop stress calculation then yields the outer diameter, which subtracted from \( r_{1x} \) results in the length of the beam. The mass of the cylinder can then also be calculated using simple geometry given the two diameters and the length.

With the individual mass estimates of all the components determined, the total mass is summed and this value is passed back up to the grid-search level and associated with a particular point on the grid. When all grid points have been evaluated, it is a simple matter to choose the one with the lowest estimated mass that satisfies the kinematic constraints.

**RESULTS**

From preliminary work, realistic constraints on the linkage geometry were understood well enough that it was possible to specify a reasonable range of values for each variable with a high level of confidence that the optimal value lay in that range. However, since this set of ranges was necessarily still quite broad in order to attain this high confidence level, it was necessary to perform the optimization in two stages. The initial stage’s purpose was to identify the region of highest concentration of high-quality solutions and to verify that this region was not on the boundary of the search space in any dimension (thus strongly suggesting that the initial range was indeed broad enough). A follow-up search was then performed over a smaller range just large enough to capture this high-quality region, which allowed for increased resolution in order to pinpoint the optimal solution with a high degree of accuracy.

**Initial Search**

**Search Grid**

The search bounds and resolutions shown in Tables 3 and 4 were set for the initial broad search:

**TABLE 3: INITIAL PRIMARY SEARCH GRID**

<table>
<thead>
<tr>
<th>( r_{1x} )</th>
<th>( r_{1y} )</th>
<th>( r_3 )</th>
<th>( r_4 )</th>
<th>( \theta_{x,i} )</th>
<th>( r' ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
<td>1</td>
<td>( \pi/2 )</td>
</tr>
<tr>
<td>Max</td>
<td>3</td>
<td>7.5</td>
<td>4</td>
<td>7</td>
<td>25.4</td>
</tr>
<tr>
<td>Step</td>
<td>0.2</td>
<td>0.375</td>
<td>0.375</td>
<td>0.5</td>
<td>1.27</td>
</tr>
</tbody>
</table>

**TABLE 4: INITIAL SECONDARY SEARCH GRID**

<table>
<thead>
<tr>
<th>( \theta_{y,i} )</th>
<th>( r_c )</th>
<th>( \theta_{c,i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>(-90^\circ)</td>
<td>(-180^\circ)</td>
</tr>
<tr>
<td>Max</td>
<td>(-30^\circ)</td>
<td>8</td>
</tr>
<tr>
<td>Step</td>
<td>2^\circ</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Recall that to scale the secondary using energy methods, as well as to evaluate the mass of the hydraulic actuator, the hydraulic pressure and cylinder diameter must also be specified. Two pressures, 20.68 and 103.42 MPa were tried, along with cylinder diameters between 6.35 and 12.70 cm in steps of 1.27 cm. This resulted in two extra grid dimensions, but as there were only two values for pressure and six for diameter there was only a twelve-fold increase in computation time, which had been allowed for when setting the values in Tables 3 and 4.

**Solutions Obtained**

In total, there were 4,756,752 primary sub-linkages and 90,272 secondary sub-linkages evaluated. The initial evaluation of candidate sub-linkages to screen out those that did
not meet the kinematic performance constraints, or that had impossible geometry (e.g. one four-bar link longer than the sum of the other three), was performed at the sub-linkage level without scaling and re-combining. This resulted in only 107,953 primaries and 962 secondaries making the cut (when you design linkages at random most aren’t very good). These statistics indicate the utility of the decomposition into sub-linkages. If this were not done it would effectively mean all permutations of the 4,756,752 primaries and 90,272 secondaries would have to be evaluated (that’s about $4.3 \times 10^{55}$ candidate linkages).

With the sub-linkage method, only the permutations of the sub-linkages that passed the first kinematic screening needed to be evaluated, reducing the number of permutations to about $1 \times 10^8$, which combined with the factor of 12 for different pressures and diameters comes to about $1.2 \times 10^9$. Of these permutations, most resulted in high levels of mechanical advantage variation when the two sub-linkages were recombined. Of the $1.2 \times 10^9$ permutations, only about $2.2 \times 10^7$ resulted in mechanical advantage variation under the 15% cutoff point.

This “small” subset of $2.2 \times 10^7$ candidate linkages had now cleared all the kinematics hurdles, and estimated mass was the only remaining criterion to be employed. The search had been parallelized into four parts both to reduce computation time and since doing it all at once would produce enough results to result in a memory overrun. The four parts were determined by the permutations of low or high pressure (20.68 or 103.42 MPa) and small or large cylinder diameter (6.35-8.89 or 10.16-12.7 cm diameter). It turned out that none of the results outside the high pressure, small diameter search section were estimated at under 15 kg mass, and in fact this section was more favorable in general by a wide margin.

**Follow-Up Search**

Using the insight gained from the initial search, a follow-up search was then conducted using a smaller and more refined grid based on the locations of the best solutions. Only the high pressure of 103.42 MPa was used, and only 5.08 and 6.35 cm cylinder diameters were checked.

**Search Grid**

The search bounds and resolutions shown in Tables 5 and 6 were used:

### Table 5: Follow-Up Primary Search Grid

<table>
<thead>
<tr>
<th>$r_{1x}$</th>
<th>$r_{1y}$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$\theta_{2,i}$</th>
<th>$r_j'(\text{cm})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>0</td>
<td>0.375</td>
<td>0.625</td>
<td>0.5</td>
<td>$4\pi/5$</td>
</tr>
<tr>
<td>Max</td>
<td>1.2</td>
<td>3.75</td>
<td>2</td>
<td>4</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Step</td>
<td>0.1</td>
<td>0.1875</td>
<td>0.125</td>
<td>0.25</td>
<td>$\pi/20$</td>
</tr>
</tbody>
</table>

### Table 6: Follow-Up Secondary Search Grid

<table>
<thead>
<tr>
<th>$\theta_{B,i}$</th>
<th>$r_c$</th>
<th>$\theta_{C,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>$-50^{\circ}$</td>
<td>-106$^{\circ}$</td>
</tr>
<tr>
<td>Max</td>
<td>$-30^{\circ}$</td>
<td>$-82^{\circ}$</td>
</tr>
<tr>
<td>Step</td>
<td>$0.5^{\circ}$</td>
<td>0.125</td>
</tr>
</tbody>
</table>

**Solutions Obtained**

Although the search bounds were narrower, the resolutions were significantly finer than in the initial search. 1,333,800 primary sub-linkages and 26,117 secondary sub-linkages were evaluated this time. Of these, 143,530 primaries and 6,604 secondaries survived the initial kinematics screening. Note that although the total number of sub-linkages searched was several times smaller than the initial search, the number that passed the initial screening was actually greater. This is due to the fact that the region selected for the follow-up search had a higher density of high-quality solutions. Recall that the identification of this region was the main goal of the initial search, as the resolution was initially too coarse to be sure that the best solution found was not significantly inferior to another undiscovered solution lying somewhere between the widely spaced grid points.

The 143,530 primaries and 6,604 secondaries, with only a single pressure and two cylinder diameters to check, resulted in $7.6 \times 10^9$ permutations, of which only about $8.1 \times 10^6$ resulted in mechanical advantage variation under the 15% cutoff point. Of these, the best solutions for 5.08 cm and 6.35 cm cylinder diameters are given in Table 7:
TABLE 7: OPTIMAL SOLUTIONS

<table>
<thead>
<tr>
<th></th>
<th>Estimated Mass (kg)</th>
<th>M.A. Variation</th>
<th>Jaw Rotation</th>
<th>Jaw Tip Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. Allowable</td>
<td>n/a 15%</td>
<td>40°</td>
<td>20°</td>
<td></td>
</tr>
<tr>
<td>5.08 cm</td>
<td>11.16 9.95%</td>
<td>38°</td>
<td>8°</td>
<td></td>
</tr>
<tr>
<td>6.35 cm</td>
<td>11.65 8.79%</td>
<td>39°</td>
<td>9.5°</td>
<td></td>
</tr>
</tbody>
</table>

Although the 5.08 cm solution is slightly better based on this data, the 6.35 cm solution was chosen as the 5.08 cm solution’s very narrow cylinder required an inconveniently long piston stroke.

Final Solution

Full information on the geometry of the final solution is now presented. Table 8 shows the values of the primary geometry, both in normalized units of $r_2$ lengths and in real units of cm. For this solution the conversion factor (i.e. the length of $r_2$ in cm) was 8.41. The last column, $x$, is the distance along the $R_3$ vector to the base of the jaw vector.

TABLE 8: FINAL SOLUTION PRIMARY DIMENSIONS

<table>
<thead>
<tr>
<th></th>
<th>$r_{1x}$</th>
<th>$r_{1y}$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$\theta_{2,l}$</th>
<th>$\theta_3$</th>
<th>$r'_i$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_2$</td>
<td>0.8</td>
<td>1.125</td>
<td>1</td>
<td>1.25</td>
<td>1.75</td>
<td>$19\pi/20$</td>
<td>$2\pi/5$</td>
<td>1.81</td>
<td>1.03</td>
</tr>
<tr>
<td>cm</td>
<td>6.73</td>
<td>9.45</td>
<td>8.41</td>
<td>10.49</td>
<td>14.71</td>
<td>$19\pi/20$</td>
<td>$2\pi/5$</td>
<td>15.24</td>
<td>8.66</td>
</tr>
</tbody>
</table>

Table 9 shows the values of the secondary geometry, both in normalized units of $r_B$ lengths and in real units of cm. For this solution the conversion factor (i.e. the value of $r_B$ in cm) was 7.72. The first column, $r_A$, overconstrains the linkage but is included for reference. Also included is $\theta_{\text{crank}}$, the angle spanned by the bell crank between $R_3$ in the primary and $R_B$ in the secondary, which is a dependent variable determined by the start angles of the two sub-linkages.

TABLE 9: FINAL SOLUTION SECONDARY DIMENSIONS

<table>
<thead>
<tr>
<th></th>
<th>$(r_A)$</th>
<th>$r_B$</th>
<th>$r_C$</th>
<th>$\theta_{B,l}$</th>
<th>$\theta_{C,l}$</th>
<th>$\theta_{\text{crank}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_B$</td>
<td>0.757</td>
<td>1</td>
<td>1.5</td>
<td>$-30.5^\circ$</td>
<td>$-94^\circ$</td>
<td>39.75°</td>
</tr>
<tr>
<td>cm</td>
<td>5.84</td>
<td>7.72</td>
<td>11.58</td>
<td>$-30.5^\circ$</td>
<td>$-94^\circ$</td>
<td>39.75°</td>
</tr>
</tbody>
</table>

Figure 19 and 20 show the closed and open positions of the half-linkage respectively. They are to scale, with the axes in cm and the top left ground pivot at the origin. The hydraulic actuator is not shown, in favor of the equivalent slot used in Figure 12. The symmetry line of the full linkage is also shown.
DISCUSSION

Kinematics

Examination of these figures shows that the end of the connecting link moves in a perfect vertical path, the bell crank rotates 120°, and that the jaw tip starts at \( x = 0 \) cm and finishes at \( x = 22.86 \) cm, all in accordance with the design requirements. Looking at the y axis values, we also see that the jaw tip moves in a very nearly straight line as the net result of a decidedly nonlinear overall motion of the linkage.

We can also note a number of qualitative properties of the final solution that indicate that the grid-search optimization has come up with an intuitively plausible result for the most mass-efficient linkage:

- The jaw is short relative to the rest of the linkage (compare to Figure 1), and the majority of the tip travel comes from translation rather than rotation of the jaw. As explained in the Jaw Motion section, this tends to reduce jaw mass.
- The bell crank spans a small angular distance (since the start angles of the secondary and primary sides of the bell crank happen to be close to each other), reducing the mass of this link.
- The two two-force members, the connecting link \((r_c)\) and the link connecting the jaw to the bottom right ground pivot \((R_4)\), are the two longest links. Two-force members are naturally much lighter for their length since they are always loaded axially, meaning the optimization has correctly favored reducing the size of three-force members.

Finally, it is worth mentioning that it is very difficult to find a linkage geometry where the bell crank can rotate 120 degrees and the connecting link’s lower end stays close to centerline (to avoid making the piston crossbar too massive) that does not at either the start or end of the motion produce transmission angles smaller than the 30° limit. Note in Figure 20 that the final transmission angle is exactly at the limit, and any more bell crank rotation would result in failure. The optimization was able to find a solution that met this difficult condition while still retaining a low-mass secondary geometry.

Considering the quantitative and qualitative data given above, it is evident that the grid search dimensional optimization method was successful in meeting the kinematic requirements.

Structural

Although the detailed mechanical design of the actual mechanism based on the final solution is not discussed, it must be noted that the design process revealed some specifics of the mass minimization model to be inadequate. As previously noted, the two linkage halves overlap in-plane and must thus operate in adjacent planes. This was not accounted for in the structural model. Detailed FEA analysis revealed that this resulted in the creation of unexpectedly large moments about the y axis, requiring significant increases in material mass to withstand. Additionally, the simple heuristics used did not in all cases adequately predict link masses. As a result, a refined mass estimate based on detailed CAD and FEA analysis was about 25 kg, roughly double that obtained from the optimization. This does not necessarily invalidate the optimality of the solution, as the same effects that resulted in underestimated mass would be present (admittedly in varying degrees) in competing solutions as well.

Computational Efficiency

As was previously mentioned, the grid search method used here is computationally expensive compared to other methods such as gradient descent or metaheuristic (e.g. genetic or particle-swarm) algorithms. Both the initial and follow-up searches took about 24 hours to run when parallelized over three Intel i5 processors. It was felt, however, that this amount of computer time was well worth the advantages of getting a complete picture of the solution space and having a very high chance of approximately locating the global minimum. Hybrid search strategies are also possible, e.g. initializing a number of gradient descent searches at each point in a much coarser grid to maximize the probability of finding the global minimum while reducing computation time as compared to a pure grid search, although this technique would not give as complete a picture of the solution space.

Future Work

The method presented here was highly successful in obtaining a satisfactory kinematic solution, and partly effective in optimizing the mass of the linkage. Future work will include addressing the deficiencies discussed above through changes in linkage topology and refinement of the structural modeling.

Now that the large impact of in-plane overlap is understood, it may be possible to come up with a revised ten-bar where the two halves do not overlap. Additionally, this method can also
be applied to the traditional six-bar topology, which does not have this problem.

In place of the simple heuristics which were found to be inadequate in certain cases, a more sophisticated method of estimation using closed-form stress analysis of more complex (but still simplified) models of the various links would likely achieve more accurate results. In the case of the six-bar, the reduced linkage complexity would exponentially reduce the size of the grid, perhaps even allowing the integration of low-end FEA to become computationally feasible.

Because the methods of evaluating the kinematic and structural properties of a candidate mechanism are decoupled and performed sequentially, this method can easily be generalized to a variety of similar problems in mechanism design that do not share the same linkage topology or even optimization goals. It would also be possible to use a multi-dimensional optimization algorithm such as a gradient descent approach in place of the computationally expensive grid search. In fact, any method that optimizes inputs based on evaluation of associated outputs is in principle compatible with the approach presented in this paper.

ACKNOWLEDGEMENT
This work was sponsored by DARPA under award W31P4Q-11-C-0060.

REFERENCES