Energy Management Strategy for a Power-Split Hydraulic Hybrid Vehicle Based on Lagrange Multiplier and Its Modifications

Zhekang Du, Kai Loon Cheong and Perry Y. Li

Abstract—Lagrange multiplier approach is a computationally efficient method for computing optimal energy management strategy for a hydraulic hybrid vehicle under the assumption that the accumulator dynamics can be ignored and only the net use of storage energy is considered. Although it provides a close estimate to the fuel economy compared to that obtained using dynamic programming, the resulting control strategy does not respect the physical limits of the storage capacity of the hydraulic accumulator. Thus, the synthesized control strategy is not feasible for actual driving. This paper investigates the basic Lagrange multiplier approach for real time control and proposes modifications so that the storage capacity is respected. It is shown that the Lagrange multiplier can be interpreted as an equivalent loss factor which turns out to be the marginal loss associated with the discharge of stored energy. The two proposed modifications are: 1) a moving horizon approach, and 2) making the Lagrange multiplier a function of the current state-of-charge. Both methods are successful in maintaining the accumulator state-of-charge within limits with modest effect on fuel economy (3-5% lower).

Index Terms—Hydraulic hybrid vehicles, energy management, dynamic programming, equivalent consumption minimization strategy (ECMS), constrained optimization, power-split vehicles

I. INTRODUCTION

Hybrid vehicles are equipped with an energy storage and regeneration device in addition to an internal combustion engine. This increases fuel economy by 1) enabling the engine to operate at higher efficiency operating points that would normally be inconsistent with the power required; 2) capturing and regenerating braking energy that would otherwise be wasted; and 3) turning off the engine when not needed. In order to take advantage of these opportunities, it is essential to manage the energy storage element intelligently. This is especially important for hydraulic hybrid vehicles because of the limited energy storage capacity of the hydraulic accumulators (which is 2 orders of magnitude smaller than electric batteries of the same physical size). The energy density challenge for hydraulic hybrids exists despite the order of magnitude advantage of hydraulic accumulators and pump/motors in power density over their electric counterparts.

The optimal energy management problem for hydraulic hybrid vehicles - i.e. the control of the storage energy to optimize the overall fuel economy while observing physical constraints, can be solved by dynamic programming (DP) if the drive cycle is known a-priori and the computational burden is not a concern [1], [2]. The need for knowledge of the drive cycle can be alleviated by the use of stochastic drive cycles or prediction via iterative learning [3], [4], [5] with slight decrease in fuel economy. In the absence of drive cycle knowledge, real-time implementable strategies have been extracted from the DP results [6], [7], [8], [9] manually or using neural network, machine learning techniques. Model predictive control (MPC) has also been used with no future information about the drive cycle [10], [11].

An alternative, computationally efficient, Lagrange multiplier approach has been proposed for solving a simplified optimal energy management problem [12], [13]. The method was developed originally for the rapid evaluation of different designs of hydraulic hybrid power-split vehicles so that an iterative design/sizing optimization can be achieved [14], [15]. It is similar to the Equivalent Consumption Minimization Strategy (ECMS) [16], [17] in that the resulting control law consists in a simple instantaneous optimization. Moreover, unlike DP, the Lagrange multiplier method does not require the detailed time course of drive cycle data. Instead, only aggregate statistics (i.e. relative frequencies or how often an event occurs instead of the temporal ordering) of the drive cycle points are needed. These features are advantageous for real-time implementation.

However, similar to ECMS, the control law synthesized using the Lagrange multiplier approach does not necessarily respect the storage capacity constraint, thus it cannot be used for real-time driving without modification. In the case of ECMS, modifications have been developed using adaptation or fuzzy logic [18], [19] to allow for real-time application. These energy management strategies (EMS) were focused on hybrid electric vehicles (HEV) with energy capacity constraints much less restrictive than hydraulic hybrid vehicles.

To develop a control law for real-time optimization for hydraulic hybrid vehicles, we first study the physical interpretation of the Lagrange multiplier. As will be shown in section III, the Lagrange multiplier can be interpreted as the marginal loss associated with the discharge of accumulator energy. Moreover, the control law can be instantiated with any current driving condition. This is not the case with the control law obtained with deterministic DP which is explicitly time
Based on this interpretation, we propose two modifications to the Lagrange multiplier approach so that the resulting control laws respect the storage capacity constraint. The first approach is to apply the Lagrange multiplier approach in a moving horizon manner. The constraint is that the accumulator state-of-charge ($SOC_E$) returns to some target value at the end of the horizon. A gain factor is also introduced to tune the aggressiveness of the control action to maintain the $SOC_E$.

This results in a time varying Lagrange multiplier that is dependent on the current $SOC_E$. The second approach is to optimize the Lagrange multiplier as a function of the $SOC_E$, which can be computed off-line. The resulting Lagrange multiplier value becomes dependent on the current $SOC_E$ instead of being a function of time.

The rest of the paper is organized as follows. In section II, the power-split hydraulic hybrid vehicle system used for this investigation is described. Section III presents the basic Lagrange multiplier approach. Sections IV-V present the moving horizon method and the state-of-charge dependent Lagrange multiplier method to ensure that the storage capacity constraint is respected. Section VI contains some concluding remarks.

II. SYSTEM DESCRIPTION AND CONTROL HIERARCHY

A. Hydraulic Hybrid Power-Split Architecture

Our investigation focuses on the input coupled power-split hydraulic hybrid architecture, also known as a hybridized hydro-mechanical transmission (HMT) as shown in Fig. 1. This architecture allows engine power to be transmitted mechanically or hydraulically (via the two pump/motors) to the wheel. The high pressure hydraulic accumulator is used to store the energy from braking; and to store or provide the energy when engine power is in excess or less than demand. The low pressure hydraulic accumulator is used as a reservoir. The power split device (PSD) is a planetary gear train that splits/combines power among its 3 inputs/outputs.

The optional gear $G$ changes the speed between the engine and the PSD. With the use of the hydraulic accumulator and by varying the displacements of the pump/motors, the engine can operate at an arbitrary speed and torque while satisfying the vehicle’s speed and torque demand. Specifically, the “Torquer” pump/motor (PM1) shifts the engine torque relative to the vehicle torque, whereas the “Speeder” pump/motor (PM2) shifts the engine operating speed relative to the vehicle speed. The clutch allows the drive train to be disconnected from the engine to operate in a “hydraulic-only” mode to reduce engine idling losses. In this mode, one of the pump/motors has the option to be locked up or allowed to free-wheel. For simplicity, neither the “hydraulic-only” mode nor engine shutoff is used in this paper, however. Although regenerative braking using the pump/motors is used for most cases, the architecture also allows the use of mechanical brake when excessive braking is needed. The HMT architecture is more efficient than either a series or a parallel architecture because engine operation can be optimized and power can be transmitted through the efficient mechanical transmission without going through the less efficient hydraulic pump/motors.

The vehicle, except the engine, used in this study is based on the utility vehicle chassis as described in [12], [20]. Relevant parameters are given in Table I.

B. Component models

1) Engine: A Toyota Prius engine model, as provided by the ADVISOR software [21], is used in this study. Its efficiency map is shown in Fig. 2. The engine loss $Loss_{eng}(\omega_{eng}, T_{eng})$ at a given engine speed and torque is defined as the difference between the thermal energy available in the fuel consumed and the mechanical work.
2) Pump/motors: A pair of variable displacement $D_{pm} = 28cc$ bent-axis pump/motors are used in this study. Volumetric and torque losses in each pump/motor ($i = 1, 2$) as functions of pump/motor speed ($\omega_{pm,i}$), displacement ratio ($x_i(t) \in [-1, 1]$) and pressure $P$ are provided by the manufacturer. $x_i(t)\omega_{pm,i}$ is positive in motoring mode, and negative in pumping mode. The flow and torque relationships are given by:

\[
Q_{pm,i} = \frac{x_i(t)D_{pm}}{2\pi} \omega_{pm,i} + Loss_Q(\omega_{pm,i}, x_i(t), P) \tag{1}
\]

\[
T_{pm,i} = \frac{x_i(t)D_{pm}}{2\pi} P + Loss_T(\omega_{pm,i}, x_i(t), P) \tag{2}
\]

where volumetric loss $Loss_Q$ is positive in the motoring mode and negative in the pumping mode; and torque loss $Loss_T$ takes the sign of the displacement $x_i(t)$ in the motoring mode and that of $-x_i(t)$ in the pumping mode. From these, the power loss in each pump/motor can be defined as $Loss_{pm,i}(\omega_{pm,i}, x_i(t), P)$. The overall efficiency is above 90% at some operating conditions. However, efficiency drops off significantly at other conditions such as at low displacements. The efficiency map at 17.2MPa is shown in Fig. 8-bottom.

3) Accumulator: A gas-charged accumulator is used in this study. For simplicity, it is assumed to operate isothermally. Let $P_{pr}$ be the pre-charged pressure and $V_{pr}$ be the pre-charged air volume which is also taken to be the accumulator volume. The dynamics and pressure are given by:

\[
\dot{V}_F = Q_{acc} = -(Q_{pm,1} + Q_{pm,2}) \tag{3}
\]

\[
P = P_{pr} \left( \frac{V_{pr}}{V_{pr} - V_F} \right) \tag{4}
\]

where $V_F$ is the volume of the hydraulic fluid in the accumulator, and $Q_{acc}$ is the net hydraulic flow into the accumulator. Although in reality, the pressure and volume are related dynamically due to heat transfer, these simplifying assumptions are expected to have only minor impact on the system behavior. The accumulator energy content or the state-of-charge ($SOC_E$)$^1$ as functions of liquid volume and pressure are:

\[
SOC_E(V_F) = \int_0^{V_F} P(V_F) \cdot dV_F = P_{pr} V_{pr} \ln \left( \frac{V_{pr}}{V_{pr} - V_F} \right)
\]

\[
SOC_E(P) = P_{pr} V_{pr} \ln \left( \frac{P}{P_{pr}} \right) \tag{5}
\]

Physical constraints limit the range of allowable $SOC_E$. In our case, the limitation is imposed by the allowable operating pressure ranges and the accumulator volume.

4) Gear ratios: The gear ratios in the gear box $G$, at the pump/motor shafts $R_1, R_2$, in the power split device $PSD$, and at the output $R_{out}$ are specified by the minimum operating point.

1$^1$In this paper, $SOC_E$ is used to denote the amount of energy stored instead of as a proportion of its capacity.

C. 3-Level Hierarchical Control Architecture

Although the hybrid HMT architecture allows arbitrary engine operating points while achieving the desired vehicle speed and torque, two consequences arise from this choice:

1) how much accumulator flow (or power) is consumed or stored;
2) how much power is lost in the engine and pump/motors;

With this recognition, a 3-level hierarchical control architecture [12], [13] can be defined to simplify the energy management control problem.

1) High level controller: This controller determines the net accumulator flow $Q_{acc}$ throughout the prescribed drive cycle. Since accumulator pressure $P$ can be measured, this is equivalent to determining the accumulator power $Pow_{acc} = PQ_{acc}$. This decision affects $SOC_E$ dynamically according to the accumulator dynamics Eqs.(3)-(5). The objective is to maintain $SOC_E$ within limits and to minimize system losses. The high level control is the concern of the energy management problem considered in this paper.

2) Mid-level controller: This is a vehicle level optimization. For the given accumulator power specified by the high level controller, the current vehicle speed $\omega_{veh}$, and demanded vehicle torque $T_{veh}$, the engine and pump/motor operating points that minimize the total loss is chosen. Since this is a static power-train optimization, a mid-level map can be generated off-line to reduce the computational cost in real-time or in the simulation of various high level control strategies.

3) Low level controller: This manipulates the engine input and the pump/motor displacement ratios $x_1(t), x_2(t)$ so that the desired engine operating point is achieved [20].
The mid-level controller presents the high-level controller an abstraction of the vehicle drive train as a loss function:

\[
\text{Loss}^* (\omega_{veh}, T_{veh}, P) := \min_{\omega_{eng}, \omega_{acc}} \text{Loss}_{veh}(\omega_{veh}, T_{veh}, \omega_{eng}, T_{eng}, P) 
\]

subject to achieving the specified \( Q_{acc} \). Let \( \text{Loss}^* (\omega_{veh}, T_{veh}, P) \) be the minimum vehicle power loss, optimized over all feasible engine operations, when the vehicle speed is \( \omega_{veh} \) and the vehicle torque is \( T_{veh} \) and if the accumulator pressure is \( P \) and the accumulator flow is \( Q_{acc} \).

Notice that the dimensionality of the high level energy management problem, the topic of this paper, has been reduced to 1 state only (the \( \text{SOC}_E \)).

D. Problem formulation

Let \( (\omega_{drive}(t), T_{drive}(t)), t \in [t_0, t_f] \) be the vehicle speed and torque for a particular drive cycle. Let \( \text{Loss}^*_{drive}(t, P, Q_{acc}) \) denote the loss function for the drive cycle:

\[
\text{Loss}^*_{drive}(t, P, Q_{acc}) := \text{Loss}^* (\omega_{drive}(t), T_{drive}(t), P, Q_{acc}) 
\]

\( \text{Loss}^*_{drive}(t, P, Q_{acc}) \) is the minimum vehicle power loss at time \( t \) of the drive cycle if the accumulator pressure is \( P \) and the accumulator flow is \( Q_{acc} \). By exploiting the mid and low level control as described above, the optimal control problem is to determine the input, which is the trajectory of accumulator flow \( Q_{acc}(\tau), \tau \in [t_0, t_f] \) (or equivalently accumulator power \( P_{acc} \)) to minimize the overall loss:

\[
\min_{Q_{acc}()} \int_{t_0}^{t_f} \text{Loss}^*_{drive}(t, P(t), Q_{acc}(t)) \cdot dt 
\]

subject to the accumulator dynamics Eq.s (3)-(4), the constraint that the \( \text{SOC}_E \) stays within its allowable limits:

\[
\forall t \in [t_0, t_f], \quad \text{SOC}_{Emin} \leq \text{SOC}_{E(t)} \leq \text{SOC}_{Emax} 
\]

and the initial and final \( \text{SOC}_E \) are the same:

\[
\text{SOC}_E(t_0) = \text{SOC}_E(t_f) 
\]

Since the drive-cycle is specified, minimizing loss is equivalent to maximizing the fuel economy.

In this paper, we assume that the drive cycle is the combined EPA urban and highway driving schedules (Fig. 3). The accumulator pre-charge pressure is 11.9MPa (1700psi). The accumulator is restricted to operate between 14MPa (2000psi) and 28MPa (4000psi). The initial pressure is \( P(t_0) = 15.7 \)MPa. The nominal accumulator volume (as defined by pre-charge gas volume \( V_p \)) is 38L. Thus, the allowable \( \text{SOC}_E \) range is (75kJ, 380kJ) with an available energy of 305kJ. The initial charge is \( \text{SOC}_E(t_0) = 125kJ \).

Standard dynamic programming (DP) can be used to solve this optimization problem with pressure dynamics and storage capacity limitations. Table II shows the results for various accumulator sizes. In all cases, the pre-charge pressure is 11.9MPa, and the allowed operating pressure range is between 14MPa and 28MPa. Notice that fuel economy improves as the storage capacity increases. However, the increase is minimal beyond the capacity of 161kJ and plateaus around 52.2mpg (5.41 l/100km). The optimal operation is such that the mean accumulator pressure tends to be low to reduce leakage and to allow the pump/motors to operate at high displacements. Figure 4 shows (for the 38L accumulator case) that the \( \text{SOC}_E \) is indeed kept within the desired bounds. Note that the change in mass of the accumulator is not considered in this study.

III. LAGRANGE MULTIPLIER METHOD

A. Problem Formulation

The dynamic programming approach is computationally intensive. The optimization can be significantly simplified by ignoring the restriction on the size and pressure dynamics of the accumulator, so that the operating pressure of the system is maintained at a constant value of \( P \). The problem becomes:

\[
J^* = \min_{Q_{acc}()} \int_{t_0}^{t_f} \text{Loss}^*_{drive}(t, P, Q_{acc}(t)) \cdot dt 
\]
For any \( \lambda \) adjoining the terminal constraint into the cost function with

\[
\bar{J}^\ast \left( \lambda \right) := \min_{Q_{acc}} \int_{t_0}^{t_f} \text{Loss}_{\text{drive}}^* (t, P, Q_{acc}(t)) + \lambda (P \cdot Q_{acc}(t)) \, dt
\]

where the instantaneous augmented loss is

\[
p(t, \lambda^\ast_{\text{drive}}, Q_{acc}) := \text{Loss}_{\text{drive}}^* (t, P, Q_{acc}(t)) + \lambda^\ast_{\text{drive}} (PQ_{acc})
\]

The optimal \( \lambda^\ast_{\text{drive}} \) is a constant that summarizes the drive cycle. It is a function of the distribution and statistics of the \( \omega_{\text{veh}}, T_{\text{veh}} \) in the drive cycle only but does not depend on the sequence or ordering of the drive cycle points.

Although \( p(t, \lambda^\ast_{\text{drive}}, Q_{acc}) \) in Eq.(16) is indexed by time \( t \), the control law can be instantiated with any instantaneous vehicle speed and desired torque \( \omega_{\text{drive}}, T_{\text{drive}} \). Figure 6 gives an example in which the control law determines the accumulator powers \( PQ_{acc} \) at different vehicle speeds and desired torques. If the statistics of vehicle speed/torque of a different drive cycle are the same as the assumed one but with a different ordering, the control law would still be optimal. This is in contrast to the control law obtained from deterministic DP, which will be explicitly a function of \( t \) and the current \( SOC_E \).

The control law in Eq.(16) minimizes an equivalent power loss function reminiscent of the Equivalent Consumption Minimization Strategy ECMS (such as in [16], [17] and others). The equivalent power loss function \( p(t, \lambda^\ast_{\text{drive}}, Q_{acc}) \) in Eq.(17) consists of the actual power loss in the vehicle drive-train and the equivalent loss associated with the accumulator power \( PQ_{acc} \) with \( \lambda^\ast_{\text{drive}} \) being the conversion factor. Typically, \( \lambda^\ast_{\text{drive}} < 0 \).

**B. Physical interpretation of \( \lambda^\ast_{\text{drive}} \) and \( p(t, \lambda^\ast_{\text{drive}}, Q_{acc}) \)**

To understand the meaning of \( \lambda^\ast_{\text{drive}} \), consider Eqs.(16)-(17). Let \( P_{\text{acc}} \) be the accumulator charging power. For each time \( t \), since

\[
\frac{d}{dt} P_{\text{acc}} (t, \lambda^\ast_{\text{drive}}, Q_{acc}) = 0,
\]

\[
\lambda^\ast_{\text{drive}} = - \frac{d}{dP_{\text{acc}}} \left[ \text{Loss}_{\text{drive}}^* (t, P, Q_{acc}(t)) \right]_{P_{\text{acc}}(t)}
\]

be solved as a functional. This leads back to the computational complexity in the Dynamic Programming method.

Suppose that \( \lambda^\ast_{\text{drive}} \) is the optimal \( \lambda \) that solves Eq.(15) for a given drive cycle. This provides a real-time static control law given by a one-dimensional minimization:

\[
Q_{acc}^\ast (t) = \arg \min_{Q_{acc}} p(t, \lambda^\ast_{\text{drive}}, Q_{acc})
\]

Fig. 4. Accumulator \( SOC_E \) using Dynamic Programming for a 38L accumulator. The allowable \( SOC_E \) range is [75kJ,380kJ].
whenever the optimal \( PQ_{acc}^\ast(t) \) is not at the limit of its allowable range. This shows that \( \lambda_{\text{drive}}^\ast \) is the additional or marginal power loss for discharging the accumulator and \(-\lambda_{\text{drive}}^\ast\) is the marginal power loss for charging the accumulator. This is consistent with the Lagrange multiplier being the marginal cost for violating the constraint Eq.(13) which is the final accumulation of \( SOC_E \).

The equivalent loss function \( p(t, \lambda_{\text{drive}}^\ast, Q_{acc}) \) in Eq.(17) can then be interpreted as follows. Since the final \( SOC_E \) must return to the initial \( SOC_E \) according to Eq.(13), any decision to charge (discharge) the accumulator at the current time must consider the total loss incurred at the current time and at the other instances in the drive cycle when the energy is discharged (charged). Thus, if \( Q_{acc} > 0 \) (or \( Q_{acc} < 0 \)), the first term in Eq.(17) is the loss that occurs at the current time to satisfy the drive cycle and to charge (discharge) the accumulator, and the second term is the augmented loss, that is incurred at the other times when discharging (charging) the accumulator.

To relate Eq.(18) to the component efficiencies, consider the power flow diagram in Fig. 5. Let \( \eta_{\text{eng}} \), \( \eta_{\text{pump}} \) and \( \eta_{\text{motor}} \) be the marginal efficiencies of the engine, and of the pump/motors during pumping and motoring respectively. In a power-split vehicle, how power is split between the hydraulic and mechanical paths is also affected by the accumulator power. This relationship is determined by the mid-level optimization in Section II-C specific for the particular power-split configuration. Let \( C_{pm2} := d \text{Pow}_{pm2}/d \text{Pow}_{acc} \) and \( C_{pm1} := d \text{Pow}_{pm1}/d \text{Pow}_{acc} \) be respectively the marginal change in motoring output power and pumping input power with respect to change in accumulator power. Correspondingly, since the output power has to be maintained, the marginal mechanical power with respect to the accumulator power is \( d \text{Pow}_{mech}/d \text{Pow}_{acc} = -C_{pm2} \). \( C_{pm2} \) and \( C_{pm1} \) are related by

\[
C_{pm1} \eta_{pm1} = 1 + \frac{C_{pm2}}{\eta_{pm2}} \quad (19)
\]

By assuming that the total output work and braking input work are constant over a drive cycle, tracing, in Fig. 5, the derivatives of the losses and in the input/output works of the pump, motor and the engine with respect to changes in accumulator stored energy, and using Eq.(19), we have:

\[
\lambda_{\text{ext}}^\ast = \frac{C_{pm2}}{\eta_{pm2}} (1 - \eta_{pm2}) + C_{pm1} (1 - \eta_{pm1}) \\
+ (C_{pm1} - C_{pm2}) \left( \frac{1}{\eta_{\text{eng}}} - 1 \right) \\
- \left( \frac{1}{\eta_{\text{eng}} \eta_{pm1}} - 1 \right) - \frac{C_{pm2}}{\eta_{\text{eng}}} \left( \frac{1}{\eta_{pm2} \eta_{pm1}} - 1 \right) \quad (20)
\]

Notice that the negative of the first term refers to the loss in the pump and the engine in order to generate one unit of accumulator energy. The second term refers to loss in the pump, motor and engine due to changes in the power split. It is zero when \( C_{pm2} = 0 \) but corresponds to reduced hydraulic loss to generate accumulator energy when power is shifted to the more efficient mechanical path (\( C_{pm2} < 0 \)). The effect is amplified by the \( 1/\eta_{\text{eng}} \) factor.

\[\lambda_{\text{drive}}^\ast = -1.32\]

### Table III

<table>
<thead>
<tr>
<th>Pressure [MPa]/(psi)</th>
<th>( \lambda_{\text{drive}}^\ast )</th>
<th>Fuel Economy [mpg]/(km/100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.2(2500)</td>
<td>-1.31</td>
<td>50.0(5.55)</td>
</tr>
<tr>
<td>20.6(3000)</td>
<td>-1.31</td>
<td>48.7(5.80)</td>
</tr>
<tr>
<td>24.0(3500)</td>
<td>-1.31</td>
<td>46.1(6.13)</td>
</tr>
<tr>
<td>27.5(4000)</td>
<td>-1.29</td>
<td>43.0(6.57)</td>
</tr>
</tbody>
</table>

Fig. 6. Accumulator power commands at various vehicle speeds and torques using the optimal \( \lambda_{\text{drive}}^\ast = -1.32 \) and \( P = 17.2 \text{MPa} \). Top: contour of accumulator power commands (positive power corresponds with charging and negative power corresponds to discharging); Bottom: Drive cycle points superimposed with charging/discharging boundaries.

\[\lambda_{\text{drive}}^\ast = -1.32\]

### C. Results with Lagrange Multiplier Method

The Lagrange Multiplier approach is applied to the combined cycle as in Section II. The optimal Lagrange multiplier \( \lambda_{\text{drive}}^\ast = -1.32 \) and the estimated fuel economy at different pressures are shown in Table III. Notice that fuel economies improve with decreasing pressure as the hydraulic components become more efficient. The estimated fuel economies using the Lagrange Multiplier method are close to the DP results in Table II if an appropriate pressure is assumed. Note that the
mean pressure using DP (16.5MPa/2397psi) is lower than the lowest pressure (17.2MPa/2500psi) assumed in the Lagrange Multiplier method.

For the sample case of $P = 17.2MPa(2500psi)$, the optimal accumulator power $PQ_{\text{acc}}^*$, from Eq.(16), at each $(\omega_{veh}, T_{veh})$ is shown in Fig. 6-Top. Comparing this with the drive cycle points (Fig. 6-Bottom) reveals at which portion of the drive cycle the accumulator is charging or discharging. Note that the operating points below the charging boundary in the bottom represent braking energy that the hydraulic system cannot harvest, which is absorbed by mechanical brake. While charging and discharging occur interspersed throughout the drive cycle, the net effect during the urban portion is charging whereas the net effect during the highway portion is discharging as shown in Fig. 7. Figure 7 also shows that the Lagrange Multiplier method results in a $SOC_E$ that goes beyond the allowable range of (75kJ, 380kJ). Figure 8 shows that engine operations are concentrated at high efficiency and high power regime or regime that consumes little power; and pump/motor PM-2 operates mainly as a motor whereas PM-1 operates both as a pump and as a motor.

To check if our physical interpretation of $\lambda_{\text{drive}}^*$ in Eq.(20) is valid, $\eta_{\text{eng}}$, $\eta_{\text{pump}}$, $\eta_{\text{motor}}$ - the mean efficiencies of the engine and the pump/motors when pumping and motoring, and $C_{pm2}$ - the marginal variation of hydraulic motoring power with respect to accumulator power are determined. $C_{pm2}$ is computed by correlating the change in hydraulic motor work with change in the final accumulator $SOC_E$. They are used to compute $\lambda_{\text{drive}}^*$ according to Eq.(20) to compare with the ones obtained by solving the optimization in Eq.(15). Notice that $C_{pm2} < 0$ so that according to Eq. (20), accumulator charging has the effect of shifting power towards the mechanical power path. Because of the wide operating ranges of the components, these mean quantities do not predict the $\lambda_{\text{drive}}^*$ precisely. However, Table IV shows that the $\lambda_{\text{drive}}^*$’s can be explained by perturbing the efficiencies and marginal sensitivities by $10 - 14\%$. Additional simulations (not shown) using engine and pump/motors with constant efficiencies generate $\lambda_{\text{drive}}^*$ that matches Eq.(20) exactly.

The Lagrange multiplier method is computationally efficient and can be used to estimate the optimal fuel economy and for use in iterative design optimization [14], [15]. However, the optimization result drives the $SOC_E$ to impractically large

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**TABLE IV**

<table>
<thead>
<tr>
<th>Pressure [MPa]/[psi]</th>
<th>$\lambda_{pm2}$</th>
<th>$\eta_{\text{eng}}$</th>
<th>$\eta_{\text{pump}}$</th>
<th>$\eta_{\text{motor}}$</th>
<th>Perturbation</th>
<th>$[\min \lambda_{\text{drive}}^<em>, \max \lambda_{\text{drive}}^</em>, \lambda_{\text{drive}}^*]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.2(2500)</td>
<td>-0.140</td>
<td>0.353</td>
<td>0.881</td>
<td>0.807</td>
<td>$\pm 14%$</td>
<td>$[-3.184, -1.301, -1.317]$</td>
</tr>
<tr>
<td>20.6(3000)</td>
<td>-0.276</td>
<td>0.353</td>
<td>0.860</td>
<td>0.775</td>
<td>$\pm 10%$</td>
<td>$[-2.572, -1.122, -1.318]$</td>
</tr>
<tr>
<td>24.0(3500)</td>
<td>-0.492</td>
<td>0.353</td>
<td>0.821</td>
<td>0.727</td>
<td>$\pm 10%$</td>
<td>$[-2.290, -0.883, -1.319]$</td>
</tr>
<tr>
<td>27.5(4000)</td>
<td>-0.295</td>
<td>0.355</td>
<td>0.794</td>
<td>0.686</td>
<td>$\pm 10%$</td>
<td>$[-2.666, -1.219, -1.293]$</td>
</tr>
</tbody>
</table>

Fig. 7. Accumulator $SOC_E$ using Lagrange Multiplier method ($P = 17.2MPa$). Note that violates the allowable $SOC_E$ range of [75kJ, 380kJ] is violated.

Fig. 8. Engine (top) and Pump/motors (bottom) operating points superimposed on their efficiency maps using the optimal $\lambda^* = -1.32$ and $P = 17.2MPa$. In the engine map, the teal line is the maximum torque curve. In the pump/motor map, the 1st and 3rd quadrants correspond to motoring mode while the 2nd and 4th quadrants correspond to pumping mode.
values. We wish to modify the Lagrange multiplier approach so that realistic $SOC_E$ limits can be observed.

Figure 9 shows the accumulator power maps if different $\lambda$’s are used in Eq.(16) instead of $\lambda^{*}_{drive}$. Compared with Fig. 6, $\lambda > \lambda^{*}_{drive}$ encourages accumulator discharging; and $\lambda < \lambda^{*}_{drive}$ encourages charging. This suggests that by modifying $\lambda_{drive}$ in Eq.(16) according to the current $SOC_E$, the $SOC_E$ can be controlled to avoid going beyond the limits.

IV. MOVING HORIZON LAGRANGE MULTIPLIER APPROACH

A. Formulation

One possible real time implementable algorithm is to apply the original Lagrange multiplier method over a moving and shorter horizon $[t, t + \Delta T]$ with the terminal constraint that the state-of-charge returns to some target value at the end of the horizon. Hence, the constraint Eq.(13) is replaced by:

$$\int_{t}^{t+\Delta T} P \cdot Q_{acc}(\tau)d\tau = -k \cdot [SOC_E(t) - SOC^*_E]$$

$$= -k \cdot \epsilon_{SOC_E}(t)$$  \hfill (21)

where $\epsilon_{SOC_E}(t) := SOC_E(t) - SOC^*_E$ is the $SOC_E$ error with $SOC^*_E$ being the target $SOC$ and $SOC_E(t)$ being the $SOC_E$ at the beginning of the current horizon; $k > 0$ is the $SOC_E$ error gain which controls the convergence rate towards $SOC^*_E$ (with $k = 1$, $SOC_E(t + \Delta t) = SOC^*_E$ is required at the end of the horizon).

Thus, the Lagrange Multiplier becomes:

$$\lambda^*(t) = \arg \max_{\lambda} \int_{t}^{t+\Delta T} \min_{Q_{acc}(\tau)} [Loss_{drive}^{*}(\tau, P, Q_{acc}(\tau)) + \lambda \left( P \cdot Q_{acc}(\tau) + \frac{k}{\Delta T} \epsilon_{SOC_E}(\tau) \right)] d\tau$$

$$\left[ \begin{array}{cc}
-\lambda \\
+\lambda \\
\end{array} \right] = \epsilon_{SOC_E}(t)$$  \hfill (22)

which is a time varying quantity. Notice that positive $\epsilon_{SOC_E}(t)$ increases $\lambda^*(t)$ and encourages discharging; and negative $\epsilon_{SOC_E}(t)$ decreases $\lambda^*(t)$ and encourages charging.

Define the instantaneous augmented loss as:

$$p(t, \lambda^*(t), P, Q_{acc}) := Loss_{drive}^{*}(\tau, P, Q_{acc}(\tau)) + \lambda^*(t) \left( P \cdot Q_{acc}(\tau) \right)$$

$$Q_{acc}^*(t) = \arg \min_{Q_{acc}(\tau)} p(t, \lambda^*(t), P, Q_{acc})$$  \hfill (23)

The control policy for time $t$ can again be generated by:

$$Q_{acc}^*(t) = \arg \min_{Q_{acc}(\tau)} p(t, \lambda^*(t), P, Q_{acc})$$  \hfill (24)

Similar to Eq.(16) except that the Lagrange multiplier is time varying, dependent on the $SOC_E$ error, and the distribution of the drive cycle operating points in the horizon ahead. The time dependency is due to variation in the drive cycle statistics as the horizon translates.

Although the terminal constraint in Eq.(21) is imposed at $t + \Delta T$ for the definition of $\lambda^*(t)$, in the actual application of the control law Eq.(24), $\lambda^*(t)$ is not applied over the horizon $[t, t + \Delta T]$ but is continuously changing. Thus, typically, Eq.(21) is never satisfied exactly. For this reason, it is sometimes necessary to use $k > 1$. It is also possible to generalize the bias term in Eq.(22) as a nonlinear function of $\epsilon_{SOC_E}$ to emphasize control action when the $SOC_E$ is closer to the allowable limits. This opportunity has not yet been explored.

B. Results - Moving Horizon

Figure 10 shows a sample accumulator $SOC_E$ history through the drive cycle. The result was obtained using a window size of $\Delta t = 40s$, target $SOC_E$ of $SOC^*_E = 180kJ$ (corresponds to 17.5Mpa (2550psi)), and $k = 2.5$. Notice that the $SOC_E$ lies within the allowable range of $[75kJ, 380kJ]$ at the expense of a slight decrease in fuel economy to 49.5mpg (5.71 l/100km).

Figures 11-12 show the effect of $SOC_E$ error $\epsilon_{SOC_E}$ and feedback gain $k$ on $\lambda^*(t)$. Figures 11 is computed from Eq.(22) with different $\epsilon_{SOC_E}$. It shows that increasing $\epsilon_{SOC_E}$ increases $\lambda^*(t)$, encouraging accumulator discharging.

Figure 12 is obtained from simulation runs using different $k$. It shows that a larger $k$ value introduces rapid changes $\lambda^*(t)$. This corresponds to a more aggressive control action to maintain the accumulator $SOC_E$. 
Fig. 10. Accumulator $SOC_E$ using an moving window $\lambda$ method through the drive cycle (with $\Delta T = 40s$, $SOC_E^* = 180kJ$ and $k = 2.5$). The maximum and minimum allowed $SOC_E$ are [75kJ, 380kJ].

Fig. 11. $\lambda(t)$ in the moving horizon method for different $e_{SOC_E}$ values (with $\Delta t = 100s$ and $k = 1$).

Fig. 12. $\lambda(t)$ in the moving horizon method for different $k$ (with $\Delta t = 100s$).

The effect of the window size $\Delta T$, $k$ value and the $e_{SOC_E}$ on the fuel economy, and $SOC_E$ ranges are shown in Tables V-VII. Max and Min $SOC_E$ correspond to the maximum and minimum achieved $SOC_E$. Table V shows that increasing $\Delta T$ tends to increase fuel economy but the $SOC_E$ range also increases. Table VI shows that larger $k$ decreases the $SOC_E$ range with small decrease in fuel economy. For the given choice of $SOC_E^*$ and $\Delta T$, $k > 1$ is needed to maintain the $SOC_E$ within the allowable range. With a smaller $\Delta T$, $k \leq 1$ can be used, but with adverse effect on fuel economy. This tradeoff may possibly be alleviated if a nonlinear bias function of $e_{SOC_E}$ in Eq.(22) is used.

Table VII shows, as expected, that increasing $SOC_E^*$ tends to shift the $SOC_E$ ranges by similar amounts.

### Table V

<table>
<thead>
<tr>
<th>$\Delta T$ [s]</th>
<th>MPG [mpg]/(l/100km)</th>
<th>Max. $SOC_E$ [kJ]</th>
<th>Min. $SOC_E$ [kJ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>48.4/5.84</td>
<td>259</td>
<td>119</td>
</tr>
<tr>
<td>40</td>
<td>48.5/5.82</td>
<td>253.5</td>
<td>102.5</td>
</tr>
<tr>
<td>50</td>
<td>48.6/5.81</td>
<td>281.5</td>
<td>87</td>
</tr>
</tbody>
</table>

### Table VI

<table>
<thead>
<tr>
<th>$k$</th>
<th>MPG [mpg]/(l/100km)</th>
<th>Max. $SOC_E$ [kJ]</th>
<th>Min. $SOC_E$ [kJ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>49.5/5.71</td>
<td>316</td>
<td>74.5</td>
</tr>
<tr>
<td>5</td>
<td>49/5.76</td>
<td>289.5</td>
<td>96.5</td>
</tr>
<tr>
<td>7.5</td>
<td>48.5/5.82</td>
<td>253.5</td>
<td>102.5</td>
</tr>
</tbody>
</table>

### Table VII

<table>
<thead>
<tr>
<th>$SOC_E^*$ [kJ]</th>
<th>MPG [mpg]/(l/100km)</th>
<th>Max. $SOC_E$ [kJ]</th>
<th>Min. $SOC_E$ [kJ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>48.5/5.82</td>
<td>253.5</td>
<td>102.5</td>
</tr>
<tr>
<td>220</td>
<td>47.5/5.95</td>
<td>298</td>
<td>136</td>
</tr>
<tr>
<td>260</td>
<td>46.6/6.06</td>
<td>339</td>
<td>175</td>
</tr>
<tr>
<td>300</td>
<td>45.3/6.24</td>
<td>376.5</td>
<td>220</td>
</tr>
</tbody>
</table>

### V. Lagrange Multiplier as a Function of $SOC_E$

Whereas the basic Lagrange Multiplier approach uses a constant $\lambda^*$ in the control law Eq.(16), the moving horizon Lagrange multiplier method uses a time varying $\lambda^*(t)$ that depends on the current $SOC_E(t)$ and the drive cycle information over the period $[t, t + \Delta T]$. Typically, $\lambda^*(t)$ needs to be computed in real time. To avoid this real time computational need, we propose making the $\lambda$ a constant function of the current $SOC_E$ but is not otherwise time varying.

We consider $\lambda(SOC_E)$ as a $n$-points piecewise linear function of the form:

$$\lambda(SOC_E) = a_k + \frac{SOC_E - s_k}{s_{k+1} - s_k}(a_{k+1} - a_k) \quad (25)$$

where $(s_i, a_i)$ are the coordinates of the $i$-th knot point, $i = 1, \ldots, n$; and in Eq.(25), the range $[s_k, s_{k+1}]$ is chosen such that $s_k \leq SOC_E \leq s_{k+1}$. This generates the control law:

$$Q^*_acc(t) = \arg \min_{Q_{acc}} p(t, \lambda(SOC_E(t)), Q_{acc}) \quad (26)$$

### Effect of Target $SOC_E$, $SOC_E^*$ in the Moving Horizon Method

<table>
<thead>
<tr>
<th>$SOC_E^*$ [kJ]</th>
<th>MPG [mpg]/(l/100km)</th>
<th>Max. $SOC_E$ [kJ]</th>
<th>Min. $SOC_E$ [kJ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>48.5/5.82</td>
<td>253.5</td>
<td>102.5</td>
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<tr>
<td>300</td>
<td>45.3/6.24</td>
<td>376.5</td>
<td>220</td>
</tr>
</tbody>
</table>
where the instantaneous augmented loss is

\[ p(t, \lambda(SOC_E(t)), Q_{acc}) := Loss_{drive}(t, P, Q_{acc}) + \lambda(SOC_E(t)) (P - Q_{acc}) \]  

(27)

For a given drive-cycle, values of the n knot points in Eq.(25) are optimized off-line to maximize fuel economy while ensuring that the \( SOC_E \) lies within its allowable range. Given a set of coefficients, the fuel economy is obtained by simulating Eqs.(26)-(27) together with the accumulator dynamics (3)-(4). Direct optimization, e.g. using the \texttt{fminsearch} function in MATLAB, can be used to optimize the coefficients.

The optimized \( \lambda(SOC_E) \) function (with \( n = 12 \)) for the combined EPA cycle in Fig. 3 is shown in Fig 13. Using the optimal constant \( \lambda^* = -1.32 \) in the basic Lagrange Multiplier method as reference, as the \( SOC_E \) gets closer to the lower limit (75kJ), \( \lambda \) becomes more negative to encourage charging; when the \( SOC_E \) increases towards the upper limit (380kJ), \( \lambda \) increases and saturates at -0.7 to encourage discharging. Saturation occurs because further increase in \( \lambda \) does not affect discharging pattern. The \( \lambda(SOC_E) \) function saturates towards the upper \( SOC_E \) limit but there is no saturation effect in the lower limit. The asymmetry encourages discharging in general so that pressure is kept low to reduce losses. The resulting \( SOC_E \) history is shown in Fig.14 which shows that the \( SOC_E \) is within the allowable range of [75kJ, 380kJ] and the \( SOC_E \) tends to stay in the lower range to improve fuel economy. As a result, the achieved fuel economy is 50.7mpg (5.57 l/100km) which is only 2.9% lower than the result using DP, 52.2mpg (5.41 l/100km).

VI. CONCLUSION

Representative fuel economies, \( SOC_E \) ranges and computational times using DP, basic Lagrange Multiplier approach and the two modified Lagrange multiplier approaches are summarized in Table VIII. Basic Lagrange multiplier approach achieves a fuel economy close to the true optimal result (from DP) but it does not satisfy the allowed \( SOC_E \) range. However, it is computationally efficient and generates a ECMS like control law that is simple to implement. The multiplier can be interpreted in terms of the marginal loss incurred by discharging the accumulator and is related to marginal component efficiencies and changes how power is split between mechanical and hydraulic paths. Both modified Lagrange multiplier approaches allow the \( SOC_E \) range to be maintained within allowable limits with only minor changes in fuel economy (3-5% lower). The fuel economy for the moving horizon approach is slightly worse than the optimized \( \lambda(SOC_E) \) approach because the pressure dynamics are not predicted. Computationally, the moving horizon Lagrange multiplier approach is more intensive in real time since the Lagrange multiplier \( \lambda^*(t) \) must be computed based on the current \( SOC \) and drive cycle information over a future horizon. The optimized \( \lambda(SOC_E) \) approach is less computationally intensive in real time but it must be optimized off-line.

The two modified Lagrange multiplier approaches solve the \( SOC_E \) limitation issue. However, the solution still requires deterministic drive-cycle information. A stochastic model of the drive-cycle and an on-line estimation approach may be possible avenues for alleviating the need for this information.

Although the proposed approaches are applied to a power-split hydraulic hybrid vehicles, the methodology should be applicable to other hydraulic and electric hybrid architectures as well.

VII. ACKNOWLEDGMENTS

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REFERENCES

### TABLE VIII

<table>
<thead>
<tr>
<th>Methods</th>
<th>Fuel economy [mpg(l/100km)]</th>
<th>SOCE range [kJ]</th>
<th>Computational cost [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic Programming</td>
<td>52.2/(5.41)</td>
<td>[75, 380]</td>
<td>2988</td>
</tr>
<tr>
<td>Basic Lagrange multiplier @17.2MPa (2500psi)</td>
<td>50.8/(5.56)</td>
<td>[10, 1490]</td>
<td>&lt;0.1</td>
</tr>
<tr>
<td>Moving horizon</td>
<td></td>
<td>[75, 356]</td>
<td>0.2</td>
</tr>
<tr>
<td>(A = 40s, k = 2.5, SOCE = 180kJ)</td>
<td></td>
<td>[84, 276]</td>
<td>&lt;0.05</td>
</tr>
</tbody>
</table>