Prototype Angle Domain Repetitive Control
- Affine Parameterization Approach

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Angle-domain repetitive disturbances refer to disturbances that are periodic in a generic angle variable which is monotonically increasing with time but not uniformly. This paper extends the classical prototype repetitive control methodology for time periodic disturbances to this situation. Instead of using an internal model approach to derive the control, an affine parameterization approach is adopted which reduces the control design methodology to one of estimating and canceling the disturbance. While the resulting control architectures are similar to the classical time-domain periodic case, the stability conditions are different and depend on the choice of signal norm. This necessitates an alternate compensator design approach for the non-minimum phase terms. Robust stability is also considered in the $L_2$ setting and an affine Q-filter concept is introduced to achieve robust stability.

1 Introduction

Repetitive control is a popular method for compensating for periodic disturbances or for tracking periodic trajectories by learning from previous cycles. In recent years, a new class of repetitive control problems has arisen in which the disturbances or desired trajectories are periodic in an alternate domain that is not time. A typical example is a system with a rotational element with its own dynamics. If disturbances or desired trajectories are functions of the angular position of the rotational element, they are periodic in the angle of rotation. However, only if the rotational speed is constant, the disturbances and the desired trajectories are periodic in time. Example applications include control of valve timing for engines and for hydraulic pump/motors, track keeping in disk drive during transient operation, control of banding in printers due to non-uniformity of a print drum, etc. In all these examples, the speed of the rotational element need not be constant all the time.

Historically, repetitive control has been developed from an internal model perspective [1] by embedding the characteristic polynomial of the disturbance in the denominator of the controller and by ensuring that the closed loop system is stable. A difficulty with repetitive control relative to other internal model controls is that the internal model can have a high order since the time delay can be large compared to the sampling time. Thus, designing and implementing gains for stabilizing the system by means such as pole-placement or LQ will be computationally prohibitive.

Prototype repetitive control avoids this difficulty by assuming that the plant has been sufficiently compensated by an inner loop controller, so that most of its dynamics can be cancelled out. This allows the closed loop to be stabilized using only a scalar gain in conjunction with the internal model. This greatly simplifies the control design and implementation.

With angle-domain repetitive disturbances, both time- and angle- domain dynamics are involved. Adaptive control is used in [4] to render the system as an angle-domain invariant system so that an internal model based repetitive control can be applied in the angle-domain. Recently, these systems have been treated generally as linear time varying (LTV) systems. For example, in [5, 6], the control design uses a time varying polynomial description and solves a Diophantine-like equation that can be quite computationally intensive. In [7], a robust controller is obtained for LTV systems using an iterative robust control design methodology. Interestingly, in all these works, the angle domain periodic disturbances are not treated directly. Rather, only the first few modes are used in the design.

In this paper, the affine parameterization or innovation feedback approach [2] is applied to the angle domain repetitive disturbance rejection problem to obtain a straightforward design methodology. The relationship between this approach and time-domain repetitive control is briefly discussed in [8]. In this framework, the controller design is reduced to one of estimating and canceling the angle domain periodic disturbance. For this reason, it is sometimes called disturbance observer. Estimators for invertible and non-minimum phase plants are proposed that result in con-
control structures similar to the classical time-domain repetitive control. Stability conditions are derived for various signal norms. Because these conditions are more stringent than the classical time-domain repetitive control case, alternate compensator design methodologies are needed. Moreover, a shaping filter (affine Q-filter) can be incorporated to affinely shape the complementary sensitivity to improve system robustness. This serves a similar role as the classical Q-filter in [3] but has the added advantages of affineness and unconditional nominal system stability.

The rest of this paper is organized as follow. Section 2 defines the angle domain repetitive control problem. In section 3, the affine parameterization framework for repetitive control design is introduced. Section 4 analyzes the angle delay operator in the time domain, needed for stability analysis. Angle-domain prototype repetitive controllers are presented in section 5. Robust stability is considered in section 6. Section 7 discusses how the compensators are designed for different cases. Section 8 contains simulations to illustrate the control design and results. Concluding remarks are given in section 9.

**Notation:** For a system \( G : x(\cdot) \rightarrow y(\cdot) \) that is a map between time (or angle) signals, we use the notation \( y(t) = (G[x])(t) \). A subscript \( \theta \) (e.g. \( x_\theta(\theta) \)) is used to denote an angle domain signal, a signal with no subscript (e.g. \( x(t) \)) is used to denote a time domain signal.

### 2 Angle domain periodic disturbance rejection problem

Consider a non-time domain variable \( \theta \) that we generically refer to as the (unwound) angle. We assume that \( \theta(t) \) is monotonically increasing but not necessarily uniformly with time such that

\[
\theta(t) = \omega(t) > 0; \quad \theta(0) = \theta_0
\]  

\( \theta(t) \) and \( \omega(t) \) are assumed known or measured. To ensure that the periodicity of the disturbance is captured, we assume that \( \omega(t) > 0 \) is upper and lower bounded: \( \exists \bar{\omega}, \underline{\omega}, \) such that

\[
\underline{\omega} \leq \omega(t) \leq \bar{\omega} > 0
\]

Since \( \theta(t) \) is monotone increasing, it is invertible. i.e. for every angle \( \epsilon \), we can uniquely find \( t \) s.t. \( \epsilon = \theta(t) \). Therefore a signal can be represented in terms of \( t \) or \( \theta \). A subscript \( \theta \) is used to denote the angle domain representation of a time domain signal. Hence, for a time domain signal \( x \) and an angle \( \epsilon = \theta(t) \),

\[
x(t) = x_\theta(\epsilon) = x(\theta^{-1}(\epsilon)) = x_\theta(\theta(t))
\]  

We consider a linear time invariant plant perturbed by a sandwiched angle domain periodic disturbance (Fig. 1):

\[
y = G_o[d + G_i[u]]
\]  

where \( G_o \) and \( G_i \) are linear time invariant (time domain) operators, and \( d \) is a angle domain \( \Gamma \)-periodic signal satisfying:

\[
d_\theta(\epsilon) = d_\theta(\epsilon - \Gamma)
\]

or expressed in time domain \( d(t) = d(\theta^{-1}(\theta(t) - \Gamma)) \).

**The control objective:** Design a controller to generate control input \( u \) so that \( y \rightarrow 0 \) in (4) despite the angle domain periodic disturbance \( d(t) \).

Note that trajectory tracking problems can also be addressed in this framework by taking \( G_o = 1 \) and the desired trajectory to be \(-d(t)\).

**Remark 1** If \( d \) is a time domain periodic disturbance, the sandwiched disturbance system can be transformed into a simpler input disturbance system

\[
y = G_o \circ G_i[x + u] \quad \text{with } \epsilon \quad G_o, \new
\]

with \( x \), such that \( G_i[x] = d \), being the input disturbance which is also time periodic. This transformation however is not meaningful if \( d \) is angle domain periodic since the resulting \( x \) is generally neither angle- nor time- periodic. \( G_i \) and \( G_o \) must therefore be treated independently.

### 3 Affine Parameterization Controller

Fig. 2 shows the affine parameterized control for the plant (4) in which the plant model is used to obtain a prediction of the output. Here, the feedback signal is the innovation \( \epsilon \) which is the difference between the actual output and the predicted output:

\[
\epsilon := y - (G_o \circ G_i)[u] = G_o[d]
\]
Because $\epsilon$ is directly related to the disturbance $d$ and can be used to recover it, affine parameterization approach is sometimes referred to as disturbance observer.

Instead of designing an output feedback controller directly, we design the controller $Q_o$ which acts on the innovation $\epsilon$:

$$u = -Q_o[\epsilon]$$

If we denote

$$\tilde{d} := (G_i Q_o) [\epsilon] = (G_i Q_o G_o) [d]$$

then,

$$y = (G_o(1 - G_i Q_o G_o)) [d] = G_o[d - \tilde{d}]$$

Since $G_i$ and $G_o$ are stable, the closed loop system is stable if and only if the affine parameterization controller $Q_o$ is stable [2]. Moreover, from (7), in order for $y(t) \to 0$, we must have $\tilde{d}(t) \to d(t)$. Hence $Q_o$ should estimate $d(t)$ from its input $\epsilon = G_o[d]$ and to cancel it.

When $G_o$ and $G_i$ are minimum phase, a straightforward solution to the repetitive control problem is to set:

$$\tilde{d}_0(\theta(t)) = d_0(\theta(t) - \Gamma)$$

which is accomplished with

$$Q_o = G_i^{-1} \circ e^{-s_\theta \Gamma} \circ G_o^{-1}$$

where $e^{-s_\theta \Gamma}$ is the $\Gamma$ angle delay, angle-domain operator that will be defined and analyzed in detail in the next section.

The control (9) is different from conventional disturbance observer in that it estimates the disturbance of the previous period instead of the current instant. This is advantageous since $G_i^{-1}$ and $G_o^{-1}$ are normally not causal.

The repetitive controller in (9) will be generalized in sections 5 and 6 to the structure in Fig. 4 where

$$C_o G_o = e^{-s_\theta \Gamma} Q_o$$

$$G_i C_i = e^{-s_\theta \Gamma} \tilde{B}_i Q_i$$

Here, $C_o$ and $C_i$ are causal compensators (LTI transfer functions), $\tilde{B}_i$ is the residual from canceling $G_o$, $k$ is a scalar, $e^{-s_\theta \Gamma}$ and $e^{-s_\theta \Gamma}$ are time delay operators, and $Q_o$ and $Q_i$ are shaping filters for improving robustness.

### 4 Angle Delay Operator $e^{-s_\theta \Gamma}$

Let $e^{-s_\theta \Gamma}$ be the angle-invariant, angle domain operator that represents a $\Gamma$– (angle) delay:

$$e^{-s_\theta \Gamma}[x_0](\theta) := x_0(\theta - \Gamma)$$

where $s_\theta$ is the Laplace variable for angle-domain signals. The corresponding time delay (from time $t$) to achieve a $\Gamma$ angle delay is:

$$T(t) := t - \theta^{-1}(\theta(t) - \Gamma)$$

We define the time-domain operator corresponding to $e^{-s_\theta \Gamma}$, the time-varying delay operator $e^{-sT(\cdot)}$ as:

$$e^{-sT(\cdot)}[x](t) := x(t - T(t))$$

where $T(t)$ is given by (13). Hence, the angle domain periodicity of a disturbance can be expressed in either the angle domain or in the time domain as:

$$(1 - e^{-s_\theta \Gamma})[d_0] = 0$$

$$\left(1 - e^{-sT(\cdot)}\right)[d] = 0$$

Note that since $e^{-sT(\cdot)}$ is time varying, it does not generally commute with other LTI operators.

The following theorem derives the induced norms of $e^{-sT(\cdot)}$ which are useful for stability and robustness analysis.
Theorem 1 Suppose that the angle $\Theta$ satisfies (1)-(2). The angle-domain delay operator has the following properties in the time-domain: for all time-domain signal $x(\cdot)$, for $p = 1, 2, \ldots$,

$$\frac{\omega}{\omega} \cdot ||x||_p^p \leq ||e^{-s\Theta} \cdot x||_p^p \leq \frac{\omega}{\omega} \cdot ||x||_p^p$$

(15)

$$||e^{-s\Theta} \cdot x||_\infty = ||x||_\infty$$

(16)

where $||x||_p := (\int_0^\infty x^p(t) dt)^{1/p}$ and $||x||_\infty := \sup_{t \geq 0} |x(t)|$ are the $L_p$ norm and the $L_\infty$ norm of the time domain signal $x(\cdot)$.

Moreover, the bounds are tight in that there exist an angular speed trajectory $\omega(\cdot)$ satisfying (2) and a time domain signal $x(\cdot)$ such that the bounds are achieved.

Proof: Since $y(t) = e^{-s\Theta} \cdot x(t) = x(t - T(t))$,

$$\sup |y(\cdot)| = \sup |x(\cdot)|$$

so that $||y||_\infty = ||x||_\infty$ which is (16).

Consider now $p = 1, 2, \ldots$,

$$||y||_p^p = \int_0^\infty |x^p(t - T(t))| dt = \int_0^\infty \frac{|x^p(\tau)|}{1 - \frac{dT}{dt}(t)} d\tau$$

(17)

where $\tau := t - T(t)$. From (13), we have $\Theta(t - T(t)) = \Theta(t) - \Gamma$. Differentiating this w.r.t. $t$, we get

$$\left(1 - \frac{dT}{dt}(t)\right) = \frac{\omega(t)}{\omega(t - T)}$$

(15) follows by substituting this into (17) and applying the bounds:

$$\frac{\omega}{\omega} \leq \frac{\omega(t)}{\omega(t - T)} \leq \frac{\bar{\omega}}{\bar{\omega}}$$

To show the bounds are tight, let $x(t)$ be the time-domain signal such that its angle domain representation is:

$$x_\Theta(\Theta) = \begin{cases} 1 & \text{if } 0 \leq \Theta \leq \Theta_1 \\ 0 & \text{otherwise} \end{cases}$$

(18)

where $\Theta_1 < 2\pi$ is a small number. To achieve the upper bound in (15), define $\omega(t)$ such that the duration of the pulse in $x(t)$ is the shortest while the duration of pulse in $e^{-s\Theta} \cdot x$ is the longest (see Fig. 5):

$$\omega(t = \Theta^{-1}(\Theta')) = \begin{cases} \omega & \text{if } 0 \leq \Theta' \leq \Theta_1 \\ \bar{\omega} & \Gamma \leq \Theta' \leq \Theta_1 + \Gamma \\
\text{Don't care} & \text{otherwise} \end{cases}$$

Note that $x(t) = 1$ for $t \in [0, \Theta_1/\bar{\omega}]$. Then, $y = e^{-s\Theta} \cdot x$ is:

$$y(t) = \begin{cases} 1 & \text{if } t \in \Theta^{-1}(\Gamma) + [0, \Theta_1/\bar{\omega}] \\ 0 & \text{otherwise} \end{cases}$$

so that $||y||_p^p = \Theta_1/\bar{\omega}$ and $||y||_\infty = 1$. Thus, $||y||_p^p = \Theta_1/\bar{\omega}||x||_p^p$.

To achieve the lower bound in (15), define $\Theta(t)$ with the roles of $\bar{\omega}$ and $\bar{\omega}$ swapped, such that

$$\omega(t = \Theta^{-1}(\Theta')) = \begin{cases} \bar{\omega} & \text{if } 0 \leq \Theta' \leq \Theta_1 \\ \omega & \Gamma \leq \Theta' \leq \Theta_1 + \Gamma \\
\text{Don't care} & \text{otherwise} \end{cases}$$

Over an interval of $\Theta_1/\bar{\omega}$, $x(t) = 1$, and over an interval of $\Theta_1/\omega$, $y(t) = e^{-s\Theta} \cdot x(t) = 1$. The desired lower bound is obtained by integration of $|x^p(t)|$ and $|y^p(t)|$.

Remark 2 1. Theorem 1 shows that in continuous time, for all $p = 0, 1, \ldots, \infty$ the induced $p$-norm

$$||e^{-s\Theta} \cdot x||_{L,p} = \left( \frac{\omega}{\omega} \right)^{1/p} \geq 1.$$  

In contrast, a constant time-delay operator $e^{-sT}$ has induced $p$-norm being 1 for all $p \in [0, 1, \ldots, \infty]$. This difference will have significance when establishing stability condition.

2. Using similar proof concepts, the $p$-norms of a signal represented in angle domain and in time domain can be shown to be bounded as:

$$\omega \cdot ||x||_p^p \leq ||x_\Theta||_p^p \leq \bar{\omega} \cdot ||x||_p^p$$

3. If the signal is in discrete time, the delayed signal $x(t_k - T(t_k))$ may not correspond to any past values. In this
case, a linear interpolation can be used. For example, if
t_j \leq t_k - T(t_k) \leq t_{j+1},

\[ x(t_k - T(t_k)) = e^{-sT} [x](t_k) = x(t_j)\lambda + x(t_{j+1})(1 - \lambda) \]

where \( \lambda = (t_k - T(t_k) - t_j)/T_s \) and \( T_s \) is the sampling time.

4. We will have for all discrete time sequence \( x(t_k) \),

\[ \sup_k |e^{-sT} [x](t_k)| \leq \sup_k |x(t_k)| \]

A sequence can easily be constructed so that the above is an equality. Hence, the induced \( L_\infty \) norm of \( e^{sT} \) as a mapping between discrete time sequences will also be 1.

5. Moreover, if the sampling time \( T_s \) is small compared to the variation of \( \Theta(t) \), then the induced \( L_p \) norm for the continuous time signals can be used as a close approximation of the induced \( L_p \) norm for the discrete time sequence.

5 Prototype Repetitive Control

In this section, we analyze the nominal design of the repetitive controller in Fig. 4 with relationships Eqs.(10)-(11) where \( Q_o = Q_i = 1 \). With \( G_o \) assumed invertible, cases of \( G_o \) being invertible and not invertible are investigated. The case of \( G_i \) not being invertible will be investigated in Section 7 where an approximate inverse is proposed.

The time leads in Fig. 4 can be absorbed into the feedback loop as in Fig. 6 where \( P(\tau_1, \tau_2) : x(\cdot) \mapsto y(\cdot) \) is the mixed domain-delay time-domain operator,

\[ P(\tau_1, \tau_2) := [e^{\tau_1} \circ e^{-sT} \circ e^{\tau_2}] \] (19)

It can be unpacked using (12) as:

\[ y(t) = P(\tau_1, \tau_2)[x](t) = x(\Theta^{-1}(\Theta(t + \tau_1) - \Gamma) + \tau_2) \] (20)

Fig. 6. Affine parameterized form of the angle domain repetitive controller in Fig. 4 after look aheads are absorbed.

In time domain, the controller in innovation feedback (Fig. 6) and output feedback form (Fig. 7) are given below:

**Innovation feedback:***

\[ m(t) = \left(1 - k\beta_\theta \circ P(\tau_o, \tau_o)\right)[m](t) + kC_o[x](t) = \left(1 - k\beta_\theta\right)[m_1](t) + kC_o[x](t) \]

\[ m_1(t) = m(t - T(t - \tau_o)) + kC_o[x](t) \]

\[ u_1(t) = P(\tau_i, \tau_o)[m](t) = m(t + \tau_i + \tau_o - T(t + \tau_i)) \]

\[ u(t) = -(C_i \circ P(\tau_i, \tau_o))[m](t) = -C_i[u_1](t) \]

**Output feedback:**

\[ m(t) = P(\tau_o, \tau_o)[m](t) + kC_o[y](t) = m(t - T(t - \tau_o)) + kC_o[y](t) \]

\[ u_1(t) = P(\tau_i, \tau_o)[m](t) = m(t + \tau_i + \tau_o - T(t + \tau_i)) \]

\[ u(t) = -(C_i \circ P(\tau_i, \tau_o))[m](t) = -C_i[u_1](t) \]

Because the amount of the time delay \( T(\cdot) \) associated with the \( \Gamma \) angle delay varies with time, care must be taken to absorb the time advance/delay properly.

5.1 \( G_o \) and \( G_i \) invertible

When \( G_o \) is invertible (stable and minimum phase), \( d \) can be reconstructed via \( d = G_o^{-1}[e] \), but typically non-causally. However, as \( T(t) \) is usually large compared to the
The repetitive controller in (9) can be generalized to a first order filtered version of \( d_0(\theta - \Gamma) \) in the angle domain:

\[
\tilde{d}_0(\theta) := \lambda \cdot \tilde{d}_0(\theta - \Gamma) + (1 - \lambda) \cdot d_0(\theta - \Gamma)
\]  

(22)

This results in the estimation error \( \tilde{d}_0 := \tilde{d}_0 - d_0 \) satisfying the dynamics:

\[
\tilde{d}_0(\theta) = \lambda \cdot \tilde{d}_0(\theta - \Gamma)
\]

(23)

Thus, for \( \lambda \in (-1, 1) \), the first order (angle-domain) filter is exponentially stable. Since \( G_o \) is stable, we also have, as \( t \to \infty \),

\[
y(t) = G_o[d - \tilde{d}](t) \to 0
\]

The disturbance estimator/controller in (22) corresponds to the innovation feedback form in Fig. 4 with \( k = (1 - \lambda) \in (-1, 1) \), \( \tilde{B}_o = 1 \). \( C_i \) and \( C_o \) are the causal inverses of \( G_i \) and \( G_o \) up to time delays as below:

\[
\begin{align*}
G_iC_i &= e^{-sT_i} \\
C_oG_o &= e^{-sT_o}
\end{align*}
\]

(24)

**Theorem 2** The affine parameterization controller given in Fig. 4 or Fig. 6 with \( \tilde{B}_o = 1 \), and the corresponding closed loop system with (4) is stable if and only if \( k = 1 - \lambda \in (0, 2) \). Furthermore, if \( G_i \) and \( G_o \) are invertible, and the compensators \( C_i \) and \( C_o \) satisfy (23)-(24), then the estimate of the disturbance \( \tilde{d}_0(\theta(t)) \to d_0(\theta(t)) \) and the output \( y(t) \to 0 \). The convergence is exponential in time and in angle.

**Proof:** Since the repetitive controller implements the angle invariant disturbance estimator (22), and \( y = G_o(d - \tilde{d}) \), we need only show that (22) provides asymptotic convergence of \( \tilde{d} \to d \). Since (22) is a linear angle-invariant system, we analyze it in the angle domain. The feedback loop is stable iff \( \lambda \in (-1, 1) \) or \( k \in (0, 2) \) as specified. In angle domain, the exponential convergence rate is \( \ln(\lambda)/\Gamma \). Since \( \omega(\theta) \geq \underline{\omega} \), convergence rate in time will be greater than \( \underline{\omega} \ln(\lambda)/\Gamma \). Note that as long as \( k \in (0, 2) \), the stability is independent of \( \underline{\omega}, \overline{\omega} \), the bounds on \( \theta \).

### 5.2 \( G_i \) invertible, \( G_o \) not invertible

When \( G_i \) can be perfectly cancelled out but \( G_o \) may not be, let \( C_i \) and \( C_o \) be given by

\[
\begin{align*}
G_iC_i &= e^{-sT_i} \\
C_oG_o &= \tilde{B}_o e^{-sT_o}
\end{align*}
\]

(25)

(26)

with \( \tilde{B}_o \neq 1 \) containing the non-minimum phase terms. The issue is that we cannot recover \( d \) directly from \( \epsilon \) and only \( \tilde{B}_o[d] \) can be used as input to the estimator. To proceed, we note that (22) for the invertible \( G_o, G_i \) case can be written in an error feedback form:

\[
\tilde{d}_0(\theta) := \tilde{d}_0(\theta - \Gamma) + k[d_0 - \tilde{d}_0](\theta - \Gamma)
\]

(27)

which is a \( \Gamma \)-angle periodic integrator with a \( \Gamma \)-delayed error as input. Mimicking this and using \( \tilde{B}_o[d - \tilde{d}] \) instead of \( d - \tilde{d} \), the disturbance estimator is defined as:

\[
\tilde{d}(t) := \tilde{d}(t - T(t)) + k\tilde{B}_o[d - \tilde{d}](t - T(t))
\]

(28)

\[
= ((1 - k\tilde{B}_o)[d] + k\tilde{B}_o[d]) (t - T(t))
\]

(29)

which involves both time- and angle-domain operators but uses \( \tilde{B}_o[d] \) (which is available) as input as intended. From (28) and \( d(t - T(t)) = \tilde{d}(t) \), the disturbance estimation error \( \tilde{d} := \tilde{d} - d \) satisfies:

\[
\tilde{d}(t) = (1 - k\tilde{B}_o)[d](t - T(t))
\]

(30)

The disturbance estimator in (28) or (29) are represented in Figs. 4, 6 and 7. Note that the innovation feedback controller in Fig. 6 depends on \( \tilde{B}_o \), but the output feedback controller in Fig. 7 does not.

**Theorem 3** Consider the mixed domain repetitive controller in Figs. 4, 6 or 7 and Eqs.(25)-(26) which implements the disturbance estimator (29). The disturbance estimate \( \tilde{d}(t) \) converges to \( d(t) \) and \( y(t) \) converges to 0 in the \( L_p \) sense with \( p \in [1, 2, \ldots, \infty] \) if

\[
\|1 - k\tilde{B}_o\|_{i,p} < (\omega/\overline{\omega})^{1/p}
\]

(31)

where \( \|Z\|_{i,p} \) denotes the induced \( p \)-norm of the time domain operator \( Z \).

**Proof:** From (30), the disturbance estimation error \( \tilde{d} = \tilde{d} - d \) satisfies:

\[
(1 - e^{-sT(1 - k\tilde{B}_o)})[\tilde{d}] = 0
\]

whose stability can be viewed as a feedback loop between \( e^{sT} \) and \( (1 - k\tilde{B}_o) \) (see the feedback loop in Fig. 4). From small gain theorem, the feedback loop is stable if the loop gain, as time domain operator, with the respective induced norm, is less than 1:

\[
\|e^{-sT(1 - k\tilde{B}_o)}\|_i < 1
\]

From Theorem 1, \( \|e^{sT}\|_{i,p} \leq (\omega/\overline{\omega})^{1/p} \), and \( \|e^{sT}\|_{i,\infty} = 1 \). Therefore, conditions in (31) ensures that loop gains are less than 1 in the \( p \)-norm with \( p = [1, 2, \ldots, \infty] \).
Remark 3  
1. Recall that \( \|Z\|_{2} \) is the maximum frequency response gain of \( Z \) and \( \|Z\|_{\infty} \) is the \( L_{1} \) norm of the impulse response of \( Z \).

2. The induced-\( \infty \) norm condition in Theorem 3 generalizes Theorem 2 in that \( k \in (0, 2) \) is equivalent to \( |1 - k| < 1 \) which is the \( \infty \)-norm condition in Theorem 3.

3. The \( p \)-norm stability condition for the angle domain disturbance case is stricter than that of the time-periodic disturbance case which must have \( \|1 - k\hat{B}_{o}\|_{1,p} < 1 \). However, as \( (\bar{\omega} - \omega) \to 0 \) the time domain disturbance case is recovered.

4. For \( p \neq \infty \), the \( p \)-norm condition is stricter than the case in Theorem 2 where \( G_{o} \) is invertible and \( \hat{B}_{o} = 1 \). Moreover, except with \( k = 1 \), even as \( \hat{B}_{o} \to 1 \), the stability condition does not converge to that in Theorem 2. This discrepancy is because Theorem 3 uses a mixed angle-time domain analysis, whereas Theorem 2 uses the angle-domain analysis alone.

5. If \( kC_{o} \) is designed such that \( k\hat{B}_{o} \approx 1 \), the stability condition will be satisfied for a wide range of angular velocities. As \( k\hat{B}_{o} \to 1 \), the stability condition is satisfied by arbitrary \( \bar{\omega}/\omega \).

6. \( G_{i} \) being invertible is necessary for asymptotic convergence with the current architecture. This is not the case with time-domain repetitive control since the sandwiched system can be converted into a system with periodic input disturbance as discussed in Remark 1. Non-minimum phase \( G_{i} \) can be dealt with by using an approximate inverse (see Section 7.)

For robust stability, the plant uncertainty is assumed to be given in terms of multiplicative uncertainty in the frequency domain, i.e.

\[
G_{actual}(j\omega) = G(j\omega)(1 + \Delta(j\omega)W_{u}(j\omega))
\]

where \( \Delta \) is any LTI operator such that \( \|\Delta\|_{\infty} = \sup_{\omega} |\Delta(j\omega)| < 1 \) and \( W_{u} \) is the uncertainty weighting. Since the maximum frequency gain is the induced 2-norm of the operator, such a description imposes that stability robustness must be analyzed using 2-norm of the time signals.

Let the compensators in the repetitive controller in Figs. 4 or 6 satisfy:

\[
C_{i}G_{i} = e^{-s\tau_{i}}Q_{si} \quad (32)
\]

\[
C_{o}G_{o} = e^{-s\tau_{0}}\hat{B}_{o}Q_{so} \quad (33)
\]

instead of (25)-(26). These choices define an affine Q-filter:

\[
Q_{shape} = Q_{si}Q_{so} \quad (34)
\]

Note that from the perspective of robustness, \( Q_{shape} \) can be implemented either in \( C_{i} \) or \( C_{o} \). Besides the frequency shaping function, \( Q_{i} \) and \( Q_{o} \) can also contain residuals from canceling \( G_{o} \) and \( G_{i} \), particularly those with non-minimum phase zeros.

**Theorem 4** For a given multiplicative plant uncertainty \( \Delta(j\omega)W_{u}(j\omega) \), the repetitive controller in Fig. 6 with compensators defined using (32)-(33) will be robustly stable in the \( L_{2} \) sense if \( Q_{shape} \) is stable and:

\[
\|Q_{shape} W_{u}\hat{B}_{o}\|_{\infty} < \left( \frac{\bar{\omega}}{\omega} \right)^{1/2} - \|1 - k\hat{B}_{o}\|_{\infty} \quad (35)
\]

where \( Q_{shape} \) is given in (34).
Proof: The feedback loop containing the model uncertainty is shown in Fig. 9. The output $y$ and input $x$ to the angle delay operator $e^{sT(\cdot)}$ are related by:

$$x = (1 - k\tilde{B}_o)[y] + (Q_{\text{shape}}k\tilde{B}_o W_u\Delta)[y]$$

Since $\Delta(j\omega)$ can have arbitrary phase,

$$\|x\|_2 \leq \left(\|((1 - k\tilde{B}_o))\|_{l_2} + \|((Q_{\text{shape}}k\tilde{B}_o W_u\Delta))\|_{l_2}\right)\|y\|_2$$

$$= \|((1 - k\tilde{B}_o))\|_{l_\infty} + \left(\|((Q_{\text{shape}}k\tilde{B}_o W_u\Delta))\|_{l_\infty}\right)\|y\|_2$$

Since from Theorem 1, $\|e^{sT(\cdot)}\|_{l_2} \leq \left(\frac{\omega}{\omega}\right)^{1/2}$, from small gain theorem, the feedback loop in Fig. 9 is stable if:

$$\left(\frac{\omega}{\omega}\right)^{1/2} \left(\|((1 - k\tilde{B}_o))\|_{l_2} + \left(\|((Q_{\text{shape}}k\tilde{B}_o W_u\Delta))\|_{l_2}\right)\right) < 1$$

The desired relation is a rearrangement of the above.

Remark 4 1. Because $Q_{\text{shape}}$ enters affinely in (35), $Q_{\text{shape}}$ can be computed directly to achieve robust stability and distributed in (32)-(33).
2. How $Q_{\text{shape}}$ is distributed between (32) and (33) has no effect on robust stability but can result in different performances because angle and time domain operators do not commute.
3. By setting $W_u = 0$ in (35), we see that the $L_2$ stability condition in Theorem 3 for the nominal system is a necessary condition for robust stability.
4. It also implies that a system that is stable according to Theorem 2 may not be robust to any size of plant uncertainty. An example is if $k \in (0,2)$ but $|1 - k| > (\omega/\omega)^{1/2}$.
5. The right hand side of (35) is maximized when $k\tilde{B}_o = 1$. Thus, from the perspective of robust stability, the compensator $kC_o$ should aim to approximate $G_o^{-1}$ (up to a delay).

The affine Q-filter has the advantage that the nominal system remains stable as long as the Q-filter is also stable. The affine Q-filter does not change the form of the innovation feedback controller as in Fig. 4 or Fig. 6. Only $C_o$ and $C_i$ are changed. In contrast, the structure of the output feedback controller changes from Fig. 7 to Fig. 8. Direct implementation in the innovation feedback form may have some advantage in its simplicity in the presence of affine Q-filter.

From Fig. 8, the output feedback form of the controller with the affine Q-filter is seen to reduce to the conventional Q-filter in [3] only if $k\tilde{B}_o = 1$. Conversely, unless $k\tilde{B}_o = 1$, the conventional Q-filter would shape the complementary sensitivity function in a non-affine way.

With the affine Q-filter, performance degradation is expected. The degree of performance degradation for the case when $Q_{so} = 1$ can be obtained from the following result:

Proposition 1 If the repetitive controller in Fig. 4 is designed using Eqs. (32)-(33) with $Q_{so} = 1$, then the output error in response to the angle-domain periodic disturbance $d$, is

$$y = G_o(1 - Q_s)[d]$$

Proof: This can be seen from (7):

$$y = G_o[d - G_iQ_o G_o d] = G_o[d - G_iC_i e^{sT}[d]]$$

In the absence of $Q_{si}$, the estimator (22) or (28) is designed such that $\bar{d}(t) \rightarrow d(t - T) = d(t)$. When $Q_{si}$ is present, the output of the controller is $Q_{si}[d]$. Therefore, we have

$$y = G_o[d - Q_{si}[d]] = G_o(1 - Q_s)[d]$$

Thus, $(1 - Q_s)$ plays the role of a performance filter. However, the frequency content of $d(t)$ depends both on the frequency content of $d_0(\omega)$ and on the angular velocity $\omega(t)$.

The formula for the case when $Q_{si} \neq 1$ is more complicated because we cannot use the convergence property of the disturbance estimator since the input to the filter will no longer be angle-periodic. However, it is clear that the error would be affine in $||((1 - Q_{so})[d])||$ and in $||((1 - Q_s)[d])||$.

7 Compensator Design via Approximate Inverse

The compensators $G_i$ and $C_i$ are designed according to (32)-(34) and (35) in order to satisfy Theorem 4. They would nominally be inverses of $G_i$ and $G_o$. When $G_o$ or $G_i$ is not invertible, the non-minimum phase zeros need to be compensated appropriately. In classical discrete time repetitive control, zero phase compensators are suggested [3]. For both stability (see e.g., Theorem 3) and performance (see e.g. (36) with $Q_{si}$ incorporating the input residual), it is desirable for the residual to be close to 1. Unfortunately, zero-phase compensator can only ensure,
by picking $k$ small enough, that $\|1 - k\tilde{B}_o\|_\infty < 1$ but not necessarily smaller than $(\bar{\omega}_d/\bar{\omega})^{1/p}$ or to minimize performance degradation. Here, the forward series expansion approach in [9] is proposed instead.

**Case 1: Non-minimum phase $G_o$ via Theorem 3**

Consider the discrete time system example,

$$G_o(q) = \frac{q^{-n_0}(1 - bq^{-1})}{A(q^{-1})}$$

where $b > 1$. Our objective is to define compensator $C_i$ such that $\|1 - k\tilde{B}_o\|$ is small enough to satisfy robust stability. Here, the inverse of the non-minimum phase zero term is approximated as in:

$$(1 - bq^{-1}) \left( \frac{q^2}{b^2} + q^3 \frac{b}{b^3} + \cdots \right) = 1 + \frac{q^n}{b^n} \approx (1 - bq^{-1})^{-1}$$

Then define:

$$kC_o = Q_{so} A(q^{-1}) \left( \frac{q^2}{b^2} + q^3 \frac{b}{b^3} + \cdots \frac{q^n}{b^n} \right) q^{-n} \quad (37)$$

where the $q^{-n}$ term is to keep $C_o$ causal. Hence, (33) becomes:

$$kC_o G_o = Q_{so} \left( 1 + \frac{q^n}{b^n} \right) q^{-(n_o + n)}$$

with $k\tilde{B}_o = (1 + \frac{q^n}{b^n})$ and $n_o = T_s(n_o + n)$. This way, by using larger $n$, we can make

$$\|1 - k\tilde{B}_o\|_\infty = \left\| \frac{q^n}{b^n} \right\|_\infty = \frac{1}{|b|^n}$$

arbitrarily small to satisfy the nominal stability in Theorem 3. $Q_{shape}$ can then be chosen to further satisfy the robust stability condition in Theorem 4.

Computationally, the forward series expansion can be obtained as a deconvolution. This approach is only one way of generating an acausal FIR approximation to the inverse of the non-minimum phase terms. The FIR filter design can be combined with the affine $Q$-filter design to reduce filter length. Other acausal filter design methodologies, such as those based on optimization, can also be applied.

**Case 2: Non-minimum phase $G_i/G_t$ via Theorems 2 & 4**

Even if $G_t$ and/or $G_o$ is non-minimum phase, we can still take $k\tilde{B}_o = 1$ in Eqs.(32)-(33), but consider the residuals for inverting $G_t$ and $G_o$ as part of the affine $Q$-filter. The forward series expansion approach described above can be used. If the zeros are important only at high frequencies where the plant uncertainty $W_o$ is also large, the non-minimum phase zero may not even need to be compensated. An advantage of taking $k\tilde{B}_o = 1$ and using Theorem 2 to ensure nominal stability is that the nominal system will be unconditionally stable for ALL angular speeds $\omega(t)$. Also, the feedback controller becomes Fig. 10 which will be simpler to implement as it does not involve a feedback loop.

As an example, suppose that $G_t$ and $G_o$ are discrete time LTI systems with sufficiently fast sampling. If $G_t$ is invertible and

$$G_o(q) = q^{-n_0}(1 - zq^{-1}) \quad A(q^{-1})$$

where $A(q^{-1})$ is a monic polynomial of $q^{-1}$ with stable roots and $z > 0$, we can design

$$C_i = G_i e^{-\tau_i}$$

$$C_o = Q_{so} A(q^{-1})$$

and use $Q_{shape} = Q_{so} R$ with $R$ being the residual in compensating for the non-minimum phase zero term in (35) to ensure robust stability. A similar approach can be taken if $G_t$ is not invertible.

**8 Simulations**

To illustrate the control design, we assume that the angular frequency $\omega(t) = \theta(t)$ is shown in Fig. 11 with $\bar{\omega}/\bar{\omega} = 4$. The disturbance in angle domain, $d_\theta(\theta)$, and in time domain, $d(t)$, are shown in Fig. 12.

The sandwich plant is given by two LTI discrete time systems with sampling time of 0.002s:

$$G_i(q^{-1}) = \frac{q^{-1}0.004679(1 + 0.9355q^{-1})}{(1 - 0.9048q^{-1})^2} \quad (38)$$

$$G_o(q^{-1}) = \frac{q^{-1}0.004679(1 + 2q^{-1})}{(1 - 0.9048q^{-1})^2} \quad (39)$$

so that $G_t$ is invertible but $G_o$ is non-minimum phase.

We define the compensators as:

$$C_i(q^{-1}) = \frac{(1 - 0.9048q^{-1})^2}{0.004679(1 + 0.9355q^{-1})}$$

$$kC_o(q^{-1}) = \frac{(1 - 0.9048q^{-1})^2}{0.004679} \left( -\frac{1}{8} + \frac{q^{-1}}{4} - \frac{q^{-2}}{2} \right)$$
with the input and output delays $\tau_i = T_i$ and $\tau_o = (1+3)T_i$. Because $1 - k\hat{B}_o = q^{1/8}$ (its impulse response is a single impulse), the induced $p-$norm for all $p \in [1, 2, \ldots, \infty]$ satisfy:

$$\|1 - k\hat{B}_o\|_{i,p} = \frac{1}{8} < \left(\frac{\omega}{\omega_s}\right)^{1/p}$$

so that the system is nominally $L_p$ and $L_\infty$ stable according to Theorem 3.

The controller is implemented in the innovation feedback form (Fig. 6). The performance of this nominal control are shown in Figs. 13 and 14 which show that error converges to 0 and the estimate of $d$ converges after 1-2 cycles.

This nominal control is then robustified by incorporating a 21 tap linear-phase FIR low pass affine Q-filter into $C_o$ as in (33). The frequency response of the affine Q-filter is shown in Fig. 15. The phase of the filter is accounted for by increasing the time delay to $\tau_o = (1+3+10)T_i$.

Figs. 16 and 17 show that the robustified controller
Fig. 18. Output error with simple controller in Fig. 10, \( n = 6 \) term series expansion, and affine Q-filter.

is also effective in estimating and canceling out the disturbance. Compared with Figs. 13 and 14, the error is only slightly degraded.

Finally, we consider the controller in which we compensate for the non-minimum phase zero using a 6-term forward series expansion, but take \( k\tilde{B}_o = 1 \) as in Fig. 10. The residual is lumped into the affine Q-filter in Fig. 15. Fig. 18 shows that the error decreases in 1 cycle and the residual error magnitude is smaller than the case where \( k\tilde{B}_o \neq 1 \) in Fig. 16. If a 5-term series expansion is used, the error would be slightly worse than that in Fig. 16.

9 Conclusions

In this paper, the classical prototype repetitive control scheme is extended to the angle-domain repetitive disturbance scenario. This is achieved by using an affine parameterization perspective instead of an internal model perspective to derive the controller. The design methodology is computationally straightforward. While the resulting controller structure is similar to the time-periodic case, the stability conditions are more stringent. This requires a different approach to compensating the non-minimum phase zeros than using a zero-phase error compensator. The forward series expansion is shown to be a useful means for doing so. The affine parameterization perspective also allows for the direct computation of an affine Q-filter for improving robustness.

References