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Traffic flow control in automated highway systems

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Abstract

The problem of vehicle traffic flow control in automated highway systems is analyzed. Given desired vehicular density and velocity profiles along a highway, stabilizing velocity controllers are designed so that the real vehicle traffic flow converges to the desired profiles. The controllers are derived using vehicle conservation flow models and Lyapunov stability techniques. A highway topology with a discrete number of lanes is considered. Vehicles can have different destinations and perform diverse maneuvers. The control law obtained is distributed and therefore is suited for implementation on AHS hierarchical multilayer architectures. Simulation results are presented. © 1999 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Traffic control; Velocity control; Partial differential equations; Automated guided vehicles; Lyapunov stability

1. Introduction

The concept of Automated Highway Systems (AHS) has been proposed to increase capacity and safety in current surface transportation systems (Varaiya, 1993). The increment in highway capacity in these systems is achieved by reducing the average inter-vehicle distance without reducing their average speed. This allows an increase in the traffic flow and a better use of the existing network of highways. One of the AHS architectures used in the California PATH program decomposes the AHS into five hierarchical layers (Varaiya & Shladover, 1991): network, link, coordination regulation and physical layer (see Fig. 1). This paper focuses on the link layer, whose goal is to control the traffic flow in sections or links of the highway's network.

At the link layer level the important macroscopic quantities to abstract are the aggregate vehicular density and traffic flow in different sections of the highway. In Karaaslan, Varaiya and Walrand (1990) a detailed traffic flow model based on the behavior of human drivers is presented. The authors replaced one of the terms that describe driver behavior with a control term intended to homogenize the density profile. This way the capacity of the highway can be better realized. This formulation is enough to solve the problem of tracking a uniform density profile. Chien, Zhang and Stotsky (1994) extend this problem to the tracking of an arbitrary density profile. They derive a controller that commands a desired velocity at each section of the highway such that the density of the entire highway conforms to a specified density profile. The control law requires the inversion of the traffic flow dynamics, which requires a certain traffic flow controllability condition. This condition is violated when the density in any section of the highway becomes very small. The control action at a point in the highway requires information from the entire highway. This problem is alleviated by a dynamic version of the control law that solves the matrix inversion dynamically.

Rao and Varaiya (1994) describe a link layer controller consistent with the AHS architecture (Varaiya, 1993) for a highway operating under normal conditions. The authors assume a fully automated highway. The control variables are the lane change proportions, the desired speeds and the maximum platoon size, although the only control implemented in SmartPath (Eskaï, Khorramabadi & Varaiya, 1992) is the proportion of lane changes.

In Papageorgiou, Blosseville and Hadi-Salem (1990) an on-ramp metering control for the Southern Boulevard
Peripheral of Paris is proposed. The authors use a discrete-space traffic-flow model whose parameters are also identified. Their goal is to stabilize the traffic to a desired density and flow. As in Karaaslan et al. (1990), the major problem with the absence of feedback control is that congestion and driver behavior prevent realization of the full highway capability. The authors linearize their model and apply a linear quadratic technique to determine the metering control, based on density and flow information at various positions on the highway. Simulations show that with feedback control congestion is decreased and that the highway is able to sustain an otherwise non-realizable capacity.

The approach taken in Li, Horowitz, Alvarez, Frankel and Roberston (1997) and in this paper to the problem of designing a control system for the link layer is significantly different from those mentioned in the literature above. A fully automated highway is considered and no a priori behavior of the vehicles is assumed. An important assumption is that the closed-loop dynamics at the vehicle level has a sufficiently high bandwidth, so that it can adequately track the reference velocity and lane change commands issued by the link layer. To derive the link layer controller, a spatially and temporally continuous model of the highway, obeying only the law of conservation of vehicles, is used. The spatial partition and sampling time for implementation must be determined by the bandwidth requirements of the link layer's temporal and spatial dynamics.

It is considered that for each conceivable scenario (e.g. normal traffic condition, stopped vehicle on highway, blocked or closed lane), a desired traffic condition on the highway consistent with its capability under that circumstance can be prescribed. The desired traffic condition is encoded by a pair consisting of a desired vehicular density profile and a desired velocity field. The determination of the desired traffic condition involves some form of optimization and it is not a problem pursued in this paper. Using the theory presented in Broucke and Varaiya (1996) to calculate this repertoire of traffic profiles it is possible to guarantee that the assumptions regarding the dynamics of the velocity control and the change lane commands are satisfied.

Three important additions to the results reported in Li et al. (1997) are presented in this paper. In Li et al. (1997) all vehicles were supposed to have the same final destination. The traffic control of a stretch of highway in which there are vehicles with different final destinations is now considered. Vehicles changing lanes are treated in a different way. In Li et al. (1997), the desired velocity for the two lanes involved in a lane change maneuver was assumed to be equal. Now this constraint is relaxed and the analysis allows one to command lane changes for vehicles in adjacent lanes that have different desired longitudinal velocities. In the California PATH hierarchical architecture depicted in Fig. 1, traffic is organized in platoons of closely spaced vehicles (Varaiya, 1993; Hsu, Eskafi, Sachs & Varaiya, 1991). There is an important distinction between the behavior of the leader vehicle in the platoon and the rest of the vehicles, or followers, which integrates that platoon. The traffic flow stabilization scheme presented in this paper considers this difference between the leader and the follower behavior.

The link layer control laws described below stabilize the actual density and velocity to their desired values a highway topology that considers a discrete number of lanes.

2. Discrete lanes highway

2.1. Notation

Consider the sketch of a two lanes highway that is depicted in Fig. 2. Vehicles are represented by small solid blocks. A group of closely spaced blocks represents a platoon. The first vehicle in a platoon is denoted as leader and the rest as followers (Varaiya, 1993). The gray blocks in Fig. 2 indicate a leader and white blocks correspond to followers. Vehicles demand different amounts of highway space depending on the maneuver they are performing; a leader demands more space than a follower, and a vehicle changing lanes requires space in the two lanes involved in the lane change maneuver. The dotted lines around the blocks in Fig. 2 represent these space requirements. The final destination of vehicles on a highway has a strong influence on the lane in which they are placed; vehicles whose exits are near must be placed in the right lane, while vehicles whose destinations are distant can travel in the left lane. In Fig. 2 the destination of vehicles is indicated with a number inside the vehicle block.

This complex behavior of vehicles on a highway implies that, for a proper description of the vehicle flow on
a highway, it is not sufficient to only specify the vehicular density and velocity as a function of position and time. The lane in which vehicles are placed, the type of maneuver they are performing and the final destination of their trip must also be specified. For these reasons, the density and velocity of vehicles in the highway are denoted by

\[ K_{t,x,y,m,c} \text{ and } V_{t,x,y,m,c} \]

where \( t \) is the time; \( x \) the position along the highway, \( x \in [0, L] \); \( y \) the lane on the highway, \( y \in \{1, \ldots, n_y\} \); \( m \) the vehicle maneuver, \( m \in \{1, \ldots, n_m\} \); and \( c \) the vehicle destination or color, \( c \in \{1, \ldots, n_c\} \).

In Fig. 2, \( y = 1 \) for the left lane and \( y = 2 \) for the right lane; \( m = 1 \) for a follower, \( m = 2 \) for a leader and \( m = 3 \) for a vehicle changing lane; the closest destination corresponds to \( c = 1 \) while the next destinations are marked as \( c = 2 \) and \( 3 \), respectively.

In the PATH AHS architecture, lane changes are constrained to one-vehicle platoons. Consider, for example, the vehicle that is changing lanes in Fig. 2. This vehicle forms a one-vehicle platoon \( (y = 1, m = 2) \) and therefore is able to change lane. To actually perform the lane change maneuver, the vehicle has to: (a) initiate a lane change maneuver \( (y = 1, m = 3) \), (b) finish the lane change \( (y = 2, m = 3) \) and (c) become a leader in the right lane \( (y = 2, m = 2) \).

The desired vehicular density is denoted by \( \bar{K}_{t,x,y,m,c} \) and the desired velocity, that it is assumed to be time independent, by \( \bar{V}_{t,x,y,m,c} \). The density and velocity errors are defined as

\[ \dot{K}_{t,x,y,m,c} = \bar{K}_{t,x,y,m,c} - K_{t,x,y,m,c} \]

and

\[ \dot{V}_{t,x,y,m,c} = \bar{V}_{t,x,y,m,c} - V_{t,x,y,m,c} \]

Tensor notation for summations will be used, e.g.

\[ K_{t,x,y,m,c} = \sum_{j=1}^{n} N_{t,x,y,m,c} K_{t,x,y,m,c} \]

indicates

Whenever an index is omitted in a variable, it is meant that the variable is not a function of that omitted argument.

2.2. Traffic modeling

To derive a model for a discrete lanes highway, a principle of conservation of vehicles is applied. To properly pose a model based on this principle, it is important to consider the variation of density along the highway that occurs due to the velocity of the vehicles, as well as the variations in lane, maneuver or destination that vehicles may have. If all these variations are considered, the dynamics of the vehicle density satisfies the following partial differential equation:

\[ \frac{\partial K_{t,x,y,m,c}}{\partial t} = -\frac{\partial}{\partial x} (K_{t,x,y,m,c} V_{t,x,y,m,c}) - \frac{\partial}{\partial y} (K_{t,x,y,m,c} r_{n_{y,m,c}}) \]

\[ -\frac{\partial}{\partial m} (K_{t,x,y,m,c} r_{m_{c,m,c}}) - \frac{\partial}{\partial c} (K_{t,x,y,m,c} r_{c_{m,c}}) \]

(3)

with \( r_{n_{y,m,c}} \) and \( r_{c_{m,c}} \) indicating rates of vehicles changing lane, type or color per unit time, respectively.

The term \( K_{t,x,y,m,c} V_{t,x,y,m,c} \) can be expressed as

\[ K_{t,x,y,m,c} V_{t,x,y,m,c} = V_{t,x,y,m,c} K_{t,x,y,m,c} \]

(4)

where \( V_{t,x,y,m,c} \) is a tensor so that

\[ V_{t,x,y,m,c} = V_{t,x,y,m,c} \]

\[ V_{t,x,y,m,c} = 0, \ \forall y_1 \neq y, \ m_1 \neq m, \ c_1 \neq c. \]

Define the derivative in the \( y \) generalized direction by

\[ \frac{\partial}{\partial y} (K_{t,x,y,m,c} r_{n_{y,m,c}}) = K_{t,x,y,m,c} r_{n_{y+1,m,c}} - K_{t,x,y,m,c} r_{n_{y,m,c}}. \]

or any other discrete approximation of the derivative. The derivatives in the \( m \), and \( c \) generalized directions are
defined in a similar fashion. Using these definitions it is possible to write Eq. (3) as

\[
\frac{\partial K_{i,x,y,m,c}}{\partial t} = -\frac{\partial}{\partial X} \left( V_{i,x,y,m,c} K_{i,x,y,m,c} \right) + N_{i,x,y,m,c}^y K_{i,x,y,m,c} + N_{i,x,y,m,c}^c K_{i,x,y,m,c},
\]

(5)

with \( N_{i,x,y,m,c}^y \) and \( N_{i,x,y,m,c}^c \) appropriately defined in terms of \( r_{i,x,y,m,c}^y \) and \( r_{i,x,y,m,c}^c \). There are in total \( n_y \times n_m \times n_t \) different equations of the form of Eq. (5) if all variations of lane, maneuver and color are considered. The term \( N_{i,x,y,m,c}^y \) can be interpreted as the product of a \( 1 \times n_y \) row vector and a \( n_y \times 1 \) column vector. The elements of \( N_{i,x,y,m,c}^y \) indicate the proportion of vehicles per unit time that is leaving lane \( y \) to change into any other lane. Similar interpretations can be made for the terms \( N_{i,x,y,m,c}^m \) and \( N_{i,x,y,m,c}^c \) in Eq. (5).

To better understand the role of \( N_{i,x,y,m,c}^y \), \( N_{i,x,y,m,c}^m \) and \( N_{i,x,y,m,c}^c \) in Eq. (5), consider again the lane change example in Fig. 2. Denote the time and position at which the lane change maneuver begins by \( t_a \) and \( x_a \). The transition between leader and lane change maneuver is indicated by

\[
N_{i,x_a,1,2,1} = [0, -r_a, 0],
\]

\[
N_{i,x_a,1,2,1}^m = [0, r_a, 0],
\]

where \( r_a \) is the rate at which a leader is granted permission for a lane change. The lateral motion is described by

\[
N_{i,x_a,1,3,1} = [-r_a, 0],
\]

\[
N_{i,x_a,2,3,1} = [r_a, 0],
\]

where \( t_a \) and \( x_a \) indicate the time and position in which the lateral motion begins and \( r_b \) is the rate at which the lateral motion is accomplished. Finally, the transition from lane change to leader is described by

\[
N_{i,x_a,2,3,1}^m = [0, 0, -r_b],
\]

\[
N_{i,x_a,2,2,1} = [0, 0, r_b],
\]

where \( t_b \) and \( x_b \) indicate the time and position in which the lateral motion ends and \( r_c \) is the rate at which a vehicle is granted permission to become a leader.

The previous explanation makes it clear that there are constraints in the composition of \( N_{i,x,y,m,c}^y \), \( N_{i,x,y,m,c}^m \) and \( N_{i,x,y,m,c}^c \). These constraints can be due to physical limitations, i.e., lane changes occur only between adjacent lanes, or obey architecture limitations, i.e., a follower cannot perform a lane change maneuver in the PATH AHS architecture. In most cases, an order can be assigned to lanes, colors and types in such a way that changes of lane, color of type are only allowed between adjacent members on their respective orders. This constraint implies that if the elements \( N_{i,x,y,m,c}^y \), \( N_{i,x,y,m,c}^m \) and \( N_{i,x,y,m,c}^c \) are placed as rows of matrices according to the given order, the only elements different from zero in these matrices are in the diagonal, first super-diagonal and first sub-diagonal. In addition, the algebraic sum of the terms in any column is zero. In the example of Fig. 2, the suggested order was assigned to the lanes, types and destinations, as can be checked from the values in the examples for \( N_{i,x,y,m,c}^y \) and \( N_{i,x,y,m,c}^c \).

For the analysis the same assumption as in (Li, Horowitz, Alvarez, Frankel & Roberston, 1997) is used: there exists a prescribed profile for the densities and velocities on the highway that satisfies

\[
\frac{\partial K_{i,x,y,m,c}}{\partial t} = -\frac{\partial}{\partial X} \left( \tilde{\nu}_{x,y,m,c} K_{i,x,y,m,c} \right)
\]

\[
+ \tilde{N}_{x,y,m,c} K_{i,x,y,m,c} + \tilde{N}_{x,y,m,c}^c K_{i,x,y,m,c},
\]

(6)

where \( \tilde{\nu}_{x,y,m,c} \) is defined similarly to Eq. (4). \( \tilde{N}_{x,y,m,c} \) and \( \tilde{N}_{x,y,m,c}^c \) represent the desired proportion of vehicles changing lane, type or color per unit time, respectively. Notice that \( \tilde{\nu}_{x,y,m,c} \), \( \tilde{N}_{x,y,m,c} \) and \( \tilde{N}_{x,y,m,c}^c \) do not depend on time. These desired profiles can be determined by planning controllers such as (Broucke & Varaiya, 1996), which consider the predicted inlet and outlet demands and the capacity of the highway system. The elements of \( \tilde{N}_{x,y,m,c} \), \( \tilde{N}_{x,y,m,c}^m \) and \( \tilde{N}_{x,y,m,c}^c \) have the same constraints in their composition as those imposed on \( N_{i,x,y,m,c}^y \), \( N_{i,x,y,m,c}^m \) and \( N_{i,x,y,m,c}^c \).

2.3. Error dynamics

Define

\[
\tilde{N}_{i,x,y,m,c}^y = \tilde{N}_{x,y,m,c}^y - N_{i,x,y,m,c}^y,
\]

\[
\tilde{N}_{i,x,y,m,c}^m = \tilde{N}_{x,y,m,c}^m - N_{i,x,y,m,c}^m,
\]

\[
\tilde{N}_{i,x,y,m,c}^c = \tilde{N}_{x,y,m,c}^c - N_{i,x,y,m,c}^c
\]

(7)

Then from Eqs. (1), (2), (5), (6) and (7), the error dynamics is

\[
\frac{\partial \tilde{K}_{i,x,y,m,c}}{\partial t} = -\frac{\partial}{\partial X} \left( \tilde{\nu}_{x,y,m,c} K_{i,x,y,m,c} \right)
\]

\[
-\frac{\partial}{\partial X} \left( \tilde{\nu}_{x,y,m,c} K_{i,x,y,m,c} \right)
\]

\[
+ \tilde{N}_{x,y,m,c}^y K_{i,x,y,m,c} + \tilde{N}_{x,y,m,c}^m K_{i,x,y,m,c} + \tilde{N}_{x,y,m,c}^c K_{i,x,y,m,c},
\]

(8)

where, again, \( \tilde{\nu}_{x,y,m,c} \) is defined similarly to Eq. (4).
Using Eq. (8) it is possible to define the control problem in a precise form: determine $\tilde{\varphi}_{i,x,y,m,c}$, $\tilde{N}_{i,x,y,m,c}^y$, $\tilde{N}_{i,x,y,m,c}^m$, and $\tilde{N}_{i,x,y,m,c}^c$ in such a way that the real density profile $K_{i,x,y,m,c}$ converges to the desired density profile $\tilde{K}_{i,x,y,m,c}$.

2.4. Stabilizing control laws

Define a change of coordinates that is not time dependent

$$G_{i,x,y,m,c} = A_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c},$$

where $A_{i,x,y,m,c}^{x,y,m,c}$ is the solution to

$$\frac{\partial A_{i,x,y,m,c}^{x,y,m,c}}{\partial x} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c}$$

$$+ A_{i,x,y,m,c}^{x,y,m,c} (\tilde{N}_{i,x,y,m,c}^{x,y,m,c} + \tilde{N}_{i,x,y,m,c}^{m} + \tilde{N}_{i,x,y,m,c}^{c}) = 0.$$  \hspace{1cm} (10)

The following lemma is introduced.

**Lemma 1.** The time derivative of $G_{i,x,y,m,c}$ in Eq. (9) satisfies

$$\frac{\partial G_{i,x,y,m,c}}{\partial t} = \frac{\partial}{\partial x} \left( A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} \right)$$

$$- A_{i,x,y,m,c}^{x,y,m,c} \frac{\partial}{\partial x} \left( \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} K_{i,x,y,m,c} \right) + A_{i,x,y,m,c}^{x,y,m,c}$$

$$\left\{ \left( \tilde{N}_{i,x,y,m,c}^{x,y,m,c} + \tilde{N}_{i,x,y,m,c}^{m} + \tilde{N}_{i,x,y,m,c}^{c} \right) K_{i,x,y,m,c} \right\}. \hspace{1cm} (11)$$

**Proof.** Taking time derivative of Eq. (9)

$$\frac{\partial G_{i,x,y,m,c}}{\partial t} = A_{i,x,y,m,c}^{x,y,m,c} \frac{\partial}{\partial t} \tilde{K}_{i,x,y,m,c}$$

$$= A_{i,x,y,m,c}^{x,y,m,c} \left\{ \left. \frac{\partial}{\partial x} \left( \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} \right) \right\} \right.$$ 

$$- A_{i,x,y,m,c}^{x,y,m,c} \frac{\partial}{\partial x} \left( \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} K_{i,x,y,m,c} \right) + A_{i,x,y,m,c}^{x,y,m,c}$$

$$\left\{ \left( \tilde{N}_{i,x,y,m,c}^{x,y,m,c} + \tilde{N}_{i,x,y,m,c}^{m} + \tilde{N}_{i,x,y,m,c}^{c} \right) K_{i,x,y,m,c} \right\}. \hspace{1cm} (12)$$

By Leibniz's rule

$$\frac{\partial}{\partial x} \left( A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} \right) = A_{i,x,y,m,c}^{x,y,m,c}$$

$$\frac{\partial}{\partial x} \left( \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} \right) + \frac{\partial A_{i,x,y,m,c}^{x,y,m,c}}{\partial x} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c},$$

and Eq. (12) can be rewritten as

$$\frac{\partial G_{i,x,y,m,c}}{\partial t} = - \frac{\partial}{\partial x} \left( A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} \right)$$

$$- A_{i,x,y,m,c}^{x,y,m,c} \frac{\partial}{\partial x} \left( \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} K_{i,x,y,m,c} \right)$$

$$+ A_{i,x,y,m,c}^{x,y,m,c} (\tilde{N}_{i,x,y,m,c}^{x,y,m,c} + \tilde{N}_{i,x,y,m,c}^{m} + \tilde{N}_{i,x,y,m,c}^{c}) K_{i,x,y,m,c}$$

$$+ \left\{ \frac{\partial A_{i,x,y,m,c}^{x,y,m,c}}{\partial x} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} + A_{i,x,y,m,c}^{x,y,m,c} \right\}$$

$$\left( \tilde{N}_{i,x,y,m,c}^{x,y,m,c} + \tilde{N}_{i,x,y,m,c}^{m} + \tilde{N}_{i,x,y,m,c}^{c} \right) K_{i,x,y,m,c}.$$

From Eq. (13) it follows that, if $A_{i,x,y,m,c}^{x,y,m,c}$ is set to satisfy Eq. (10), then Eq. (11) holds.

**Remark.** The solution of Eq. (10) can be computed offline, as it is not dependent on time and only depends on the desired traffic conditions.

For the longitudinal-feedback velocity term it is important to guarantee that vehicles on the same lane have the same velocity at any particular position $x$, regardless of vehicle's destination or type, i.e., it is necessary to ensure that

$$V_{i,x,y,m,c}^x = V_{i,x,y}^x, \quad \tilde{\varphi}_{i,x,y,m,c} = \tilde{\varphi}_{i,x,y}.$$ 

To allow lane change between lanes whose desired velocity is different, it is necessary to impose an additional constraint in the desired traffic flow profile. This constraint is established in the following definition.

**Definition 1.** The desired traffic flow profile represented by $\tilde{K}_{i,x,y,m,c}$, $\tilde{\varphi}_{i,x,y,m,c}$, $\tilde{\varphi}_{i,x,y,c}^c$, $\tilde{N}_{i,x,y,m,c}^m$ and $\tilde{N}_{i,x,y,m,c}^c$ is feasible if in addition to Lemma 1, the following condition is satisfied

$$A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c} > 0,$$

$$\forall \tilde{K}_{i,x,y,m,c} \neq 0. \hspace{1cm} (14)$$

Notice that Definition 2 is always satisfied when the velocity across the lanes is the same. In this case, $\tilde{\varphi}_{i,x,y} = \tilde{\varphi}_{i,x,y}$ and therefore $A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y} \tilde{K}_{i,x,y,m,c} = \tilde{\varphi}_{i,x,y} A_{i,x,y,m,c}^{x,y,m,c} \tilde{K}_{i,x,y,m,c}$.

The following control law is proposed

$$\tilde{\varphi}_{i,x,y} = - \sum_{l \in \Gamma} \Gamma_{i,x,y,m,c} \frac{\partial}{\partial x} \left( A_{i,x,y,m,c}^{x,y,m,c} \tilde{\varphi}_{i,x,y,m,c}^{x,y,m,c} \right),$$

where $\Gamma_{i,x,y,m,c} > 0$ is a gain with $\Gamma_{i,0,y,m,c} = \Gamma_{i,1,y,m,c} = 0$.

The terms $\tilde{N}_{i,x,y,m,c}^m$, $\tilde{N}_{i,x,y,m,c}^c$, and $\tilde{N}_{i,x,y,m,c}^c$ have to be constrained in such a way that only valid changes of lane, type or color are commanded. The imposed constraints
must be equal to those of \( \hat{N}_{i,x,y,m,e}^y \), \( \hat{N}_{i,x,y,m,e}^m \) and \( \hat{N}_{i,x,y,m,e}^c \). Using the same constraints will guarantee that \( \hat{N}_{i,x,y,m,e}^y \), \( \hat{N}_{i,x,y,m,e}^m \) and \( \hat{N}_{i,x,y,m,e}^c \) will have the same structure as \( \hat{N}_{i,x,y,m,e}^y \), \( \hat{N}_{i,x,y,m,e}^m \) and \( \hat{N}_{i,x,y,m,e}^c \). Therefore, no invalid behaviors will occur on the highway.

Introduce the auxiliary quantity

\[
F_{i,x,y,m,e} = A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e},
\]

and let the feedback terms \( \hat{N}_{i,x,y,m,e}^y \), \( \hat{N}_{i,x,y,m,e}^m \) and \( \hat{N}_{i,x,y,m,e}^c \) be defined by

\[
\hat{N}_{i,x,y,m,e}^y = \begin{cases} 
- \xi,_{i,x,y,m,e} F_{i,x,y,m,e} (A_{x,y,m,e}^{y,m,e} - A_{x,y,m,e}^{y,m,e}), & |y - y_i| = 1, \\
0, & \text{else}, 
\end{cases}
\]

(17)

\[
\hat{N}_{i,x,y,m,e}^m = \begin{cases} 
- \xi,_{i,x,y,m,e} F_{i,x,y,m,e} (A_{x,y,m,e}^{y,m,e} - A_{x,y,m,e}^{y,m,e}), & |m - m_i| = 1, \\
0, & \text{else}, 
\end{cases}
\]

(18)

\[
\hat{N}_{i,x,y,m,e}^c = \begin{cases} 
- \xi,_{i,x,y,m,e} F_{i,x,y,m,e} (A_{x,y,m,e}^{y,m,e} - A_{x,y,m,e}^{y,m,e}), & |c - c_i| = 1, \\
0, & \text{else}, 
\end{cases}
\]

(19)

where the gains \( \xi,_{i,x,y,m,e} \) and \( \xi,_{i,x,y,m,e} \) are non-negative.

The \( L_2 \) norm of the density error \( \hat{R}_{i,x,y,m,e} \) is defined to be

\[
\| \hat{R}_{i,x,y,m,e} \|_2^2 = \int_0^L \hat{R}_{i,x,y,m,e}^2 \, dx.
\]

The main result of this paper is stated in the following theorem.

**Theorem 1.** Consider the highway model of Eq. (3) and suppose the desired highway conditions satisfy Eq. (6) and is feasible, according to Definition 1. Assume the inlet flow condition is such that \( \hat{R}_{1,0,y,m,e} = 0 \). Then, under the control laws in Eqs. (15) and (17)–(19) the equilibrium \( \hat{R}_{i,x,y,m,e} = 0 \) and \( \hat{R}_{i,x,y,m,e} = 0 \) for all \( x \in [0, L] \) are \( L_2 \) stable.

**Proof.** Choose the following Lyapunov candidate

\[
W_{i,x,y,m,e} = \frac{1}{2} \int_0^L (A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} (A^{-1})_{x,y,m,e} (G_{x,y,m,e})^2) \, dx,
\]

(20)

where \( (A^{-1})_{x,y,m,e} \) is the inverse tensor of \( A_{x,y,m,e} \), i.e.,

\[
(A^{-1})_{x,y,m,e} A_{x,y,m,e} = \hat{R}_{i,x,y,m,e}.
\]

First, notice that by virtue of Definition 1, Eq. (20) is positive definite. This follows from

\[
A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} (A^{-1})_{x,y,m,e} (G_{x,y,m,e})^2
\]

\[= A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e} \hat{R}_{i,x,y,m,e} > 0.\]

Taking the time derivative of Eq. (20) and using Eqs. (9) and (11)

\[
W_{i,x,y,m,e} = \int_0^L A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} (A^{-1})_{x,y,m,e} (G_{x,y,m,e}) \frac{\partial G_{x,y,m,e}}{\partial t} \, dx
\]

\[= - \int_0^L A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} \frac{\partial}{\partial x} (A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e}) \, dx
\]

\[+ \int_0^L A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e} \frac{\partial}{\partial x} (\hat{P}_{x,y} K_{i,x,y,m,e}) \, dx
\]

\[+ \int_0^L A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e} \frac{\partial}{\partial x} (\hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e}) \, dx.
\]

(21)

The first term in Eq. (21) is an exact differential in \( x \) and the second can be written using Leibnitz's rule again, thus

\[
W_{i,x,y,m,e} = - \frac{1}{2} (A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e} \hat{R}_{i,x,y,m,e})_{x=0}^L
\]

\[+ \int_0^L \frac{\partial}{\partial x} (A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e}) \, dx
\]

\[+ \hat{P}_{x,y} K_{i,x,y,m,e} \, dx - A_{x,y,m,e}^{y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} K_{i,x,y,m,e} \, dx
\]

\[+ \hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e} \frac{\partial}{\partial x} (\hat{P}_{x,y} \hat{R}_{i,x,y,m,e} A_{x,y,m,e}^{y,m,e}) \, dx.
\]

(22)

Consider that \( \hat{R}_{1,0,y,m,e} = 0 \) by assumption, and choose \( \hat{P}_{1,x,y} \hat{N}_{i,x,y,m,e}^y \hat{N}_{i,x,y,m,e}^m \hat{N}_{i,x,y,m,e}^c \) according to Eqs. (15) and (17)–(19). Then \( W_{i,x,y,m,e} \) in Eq. (22) becomes negative semidefinite, i.e.

\[
W_{i,x,y,m,e} \leq 0,
\]

and therefore \( L_2 \) stability of \( \hat{R}_{i,x,y,m,e} = 0 \) follows.

Define a global Lyapunov function as

\[
W_i = \sum_{y,m,e} W_{i,x,y,m,e}
\]

(23)
From Eq. (23)

$$W_i = \sum_{j,m} W_{i,j,m} \leq 0,$$

and the $L_2$ stability of $\bar{K}_{t,x} = 0$ follows.

**Remarks.**

1. If the inlet error condition $\bar{K}_{t,x,y,m} = 0$ is not satisfied, the controller cannot guarantee density convergence, i.e. $\bar{K}_{t,x,y,m} \neq 0$. However, the controller will still achieve a profile $\bar{K}_{t,x,y,m}$ that minimizes the weighted square integral of the error.

2. The structure of the longitudinal velocity control law in Eq. (15) and the change lane, color or type control laws in Eqs. (17)-(19) guarantees that there is no control action when $\bar{K}_{t,x,y,m} = 0$. Additionally, it should be noticed from Eqs. (8) that, if there are no vehicles, i.e., $K_{t,x,y,m} = 0$, no control is applied.

3. To calculate the control laws in Eqs. (15)-(19) only local information is required. Neighbor vehicles are assumed to interchange information and the road side infrastructure is supposed to set the values for the desired density $\bar{K}_{t,x,y,m}$ and desired velocity $\bar{v}_{x,y}$. More details about the PATH architecture can be found in Varaiya (1993) and Horowitz (1997).

2.5. Output mappings

To derive measures of highway performance it is possible to operate on the highway states. For example, the total density at a given time $t$ and position $x$, $K_{t,x}$, is given by

$$K_{t,x} = \sum_{x,y,m} K_{t,x,y,m} ,$$

while highway occupancy between $x = x_i$ and $x = x_f$ at time $t$, $O_t(x_i, x_f)$, is determined from

$$O_t(x_i, x_f) = \frac{1}{x_f - x_i} \int_{x_i}^{x_f} \left( \sum_{x,y,m} K_{t,x,y,m} S_{y,m} \right) dx ,$$

where $S_{y,m}$ is the highway space used by vehicle type $l$ in lane $y$.

2.6. Simulation results

SmartPath (Eskafi et al., 1992) is a computer simulation and visualization program which is capable of simulating an entire AHS with thousands of cars. In the simulation results that follow, the complete dynamics of the coordination, regulation and physical layers of each vehicle was simulated. SmartPath simulations were performed for one and two lane highways. In both cases an oval shaped track was used. The length of the oval is approximately 5 km. There are about 100 vehicles per lane traveling at a nominal speed of 25 m/s. The circulation is in the counterclockwise direction.

The objective in the one lane simulation is to test the ability of the link layer controller to empty sections of highway. This capability is important in AHS systems because, for example, it provides space for vehicles entry to the AHS.

Figs. 3-5 show the simulation results for a one lane highway. Each block in Figs. 3-5 represents a platoon of vehicles including the headway of its leader, 60 m in these simulations. As Fig. 5 clearly illustrates, after 160 s there are large empty sections of highway in the two straight sections of the oval highway. It should be noticed that there is a reduction in the number of platoons, and therefore in the occupancy of the highway, due to the regulation and coordination layer control laws that enforce the occurrence of joins. The size of the empty sections is much larger than the one that can be obtained without the use of the link layer controller proposed here.

In Figs. 6-9 a two-lane highway simulation on SmartPath (Eskafi et al., 1992) is illustrated. The link layer controller was required to perform two different tasks. The first task is to take place in the lower straight section of the oval highway, consists of homogenizing the vehicle density on both lanes. The second task is to
empty the inner lane of the highway at the end of the upper straight section of the oval. The results in Figs. 6–9, that correspond to 0, 40, 80 and 120 s of simulation time, respectively, indicate that the link layer controller performed the two tasks successfully. The small boxes in the figures are actually platoons of vehicles (Varaiya, 1993). The length of the oval in Figs. 6–9 is approximately 5 km. Vehicles are supposed to travel at a nominal velocity of 25 m/s. The circulation is in the counterclockwise direction.

3. Conclusion

Link layer stabilizing control laws for AHS systems are presented. These laws are proposed to stabilize traffic conditions specified by velocity and density profiles. Controls are considered for a highway model with multiple lanes in which vehicles are bounded for different destinations and can perform maneuvers from a finite set. The derivation of the link layer control laws was based on stability considerations. Properties of vehicle conservation laws for highway systems were exploited to obtain remarkably simple laws, within a rigorous framework. The control law structure is simple, consisting of a feedforward action and feedback action. The feedback control law requires the generalized gradient of a weighted density error to be calculated using only local or decentralized information. Therefore, the control scheme is suitable for AHS hierarchical architectures.

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References


