1. Lyapunov and I/O Stability
Consider the scalar system \((x \in \text{Re})\)

\[
\dot{x} = \frac{-1}{1 + t} x + u
\]  
(1)

(a) With \(u(\cdot) \equiv 0\), show that the equilibrium \(x = 0\) is asymptotically stable in the sense of Lyapunov at \(t_0 = 0\) by considering its transition matrix, \(\Phi(t, 0)\). Graph \(x(t)\) when \(x(0) = 1\).

(b) With input, show that the system is not input/output stable. Design an input with \(\|u(\cdot)\|_\infty = 1\) that such that \(\|x(\cdot)\|_\infty\) is at least 100. Simulate it to verify that it is indeed so.

[Hint: Find \(t\) such that \(\int_0^t |h(t, \tau)| d\tau > 100\).]

2. Consider the double integrator system, \(y(s) = \frac{u(s)}{s^2}\).

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{align*}
\]

(a) With \(u(\cdot) = 0\), what are all the equilibrium states of this system?
(b) Are these equilibrium states Lyapunov stable or asymptotically stable?
(c) With \(u(\cdot) \neq 0\), is the system input/output stable?

3. Stability of Periodic Systems
Consider the periodic system:

\[
\dot{x} = A(t)x
\]

where \(A(t)\) is T-periodic, i.e. \(A(t + T) = A(t)\). From HW1, we know that:

\[
x(t) = \Phi(t, 0) \Phi^k(T, 0) x(0)
\]

where \(\Phi(t_1, t_0)\) is the transition matrix, \(t_1 \in [0, T)\) and \(kT \leq t < (k + 1)T\).

What conditions should the eigenvalues of \(\Phi(T, 0)\) satisfy for the equilibrium \(x = 0\) to be: a) Lyapunov stable; b) asymptotically stable?
4. (Weighted least squares state transfer problem)
   
   (a) Let \( L_r \in \mathbb{R}^{n \times m} \) with \( \text{rank}(L_r) = n \) and \( m \gg n \). Use the geometric method (orthogonal decomposition and orbital diagram) to derive the weighted least norm solution \( U \) to:
   
   \[ x_1 = L_r \cdot U, \]
   
   for a given \( x_1 \in \mathbb{R}^n \) that minimizes the weighted cost
   
   \[ J(U) := U^T R U \]
   
   where \( R \in \mathbb{R}^{m \times m} \) is a positive definite weighting matrix.
   
   (b) Consider the double integrator system:
   
   \[ \dot{x} = Ax + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
   
   Compute (using Matlab) and plot the optimal input \( u(\tau \in [0, 10]) \) and the associated state trajectory \( \tau \in [0, 10] \) that transfers the state from
   
   \[ x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x(10) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \]
   
   and minimizes the weighted input,
   
   \[ J(u(\cdot)) = \int_0^{10} \left[ 1 + \frac{r^2}{25} \cdot (\tau - 5)^2 \right] \| u(\tau) \|^2 d\tau. \]
   
   Thus, the cost of control in the middle of the period is cheaper. Try the cases \( r = 0 \) and \( r = 5 \). [Note: You will have to do numerical integrations. Sample time of 0.01s should be sufficient if you use Euler 1-step integrator.]

5. (Receding Horizon Least Norm Transfer) The solution for problem 4 is in an open loop control structure, and thus is not robust to disturbances etc.
   
   Suppose that the state \( x(t) \) is measured at time \( t \). The sensible thing to do is at time \( t \) to re-compute the optimal control law for \( \tau \in [t, T] \) (where \( T = 10 \) is the final time) given this piece of information, and apply the control \( u(\tau = t) \). Derive the recursive control law of the form:
   
   \[ u(t) = \alpha(t) x_f + \beta(t) x(t) \]
   
   Find \( \alpha(t) \) and \( \beta(t) \) and simulate the results. Try to add some noise to your control input to simulate disturbance. You can compute \( \alpha(t) \) and \( \beta(t) \) beforehand and save them in memory if you want. What issues do you anticipate when \( t \) approaches \( T \)?
   
   If interested, instead of using the system in Problem 4, you may consider the Papi-Rubber system you derived in HW1. In this case, let Papi’s initial position be \((0, 0)\), spiky ball’s initial position be \((1, 0)\) and initial velocity be \((0, 0)\). Your control goal is to hit the target at \((-1, 0)\) at \( t = 5s \) (you may change this). The cost can be:
   
   \[ J(u(\cdot)) = \int_0^5 \left[ 1 + \frac{r^2}{25} \cdot (\tau - 5)^2 \right] (\dot{x}^2 + \dot{y}^2) d\tau. \]
[Hint 1: Re-derive the open loop least norm control problem with initial condition \( x(t) \) first]

[Hint 2: You may need to run the system open loop for the last 0.5 or 1s of the simulation to avoid your Grammian becoming singular].

6. (Controllability tests) Use the controllability matrix test and the PBH test to check if the following systems are controllable.

\[
\dot{x} = Ax + Bu
\]

a) \( A = \begin{pmatrix} 1.5 & 0.5 & -0.5 \\ -0.5 & 2.5 & 0.5 \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \)

b) \( A = \begin{pmatrix} 1.5 & 0.5 & -0.5 \\ -0.5 & 2.5 & 0.5 \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}. \)

7. (Hautus test and modal decomposition) The Hautus tests for controllability / observability may appear a bit magical to you. The conditions become very obvious for single input single output systems and with \( A \) simple. If \( A \) is simple, we can define a new set of coordinates so that \( g = T^{-1}x \), where \( T \) consists of the columns of the independent eigenvector set; and

\[
\dot{g} = \Lambda g + \bar{B}u; \quad y = \bar{C}g \tag{2}
\]

where \( \bar{B} = T^{-1}B, \bar{C} = CT \) and \( \Lambda \) is diagonal with distinct diagonal elements.

- Apply the PBH controllability and observability tests to the transformed system (2) to obtain conditions on \( \bar{B} \) and \( \bar{C} \) for the system to be controllable / observable.

- Discuss what \( \bar{B} \) and \( \bar{C} \) must be for the system to be controllable and observable, if \( \bar{A} \) is in the general Jordan form, e.g.

\[
\bar{A} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\]

- If \( A \) is semi-simple with one repeated eigenvalue, can a single input single output system be controllable and / or observable? Explain.