1. (Modeling) In the Papi-rubber game (Fig. 1), the player moves the red Papi which in turns move a spiky ball via a rubber band. The objective is to hit the aliens before they hit (you) the Papi. In this problem, we model the dynamics of the Papi and the spiky ball.

We assume that the control input is the velocity (2-D) of Papi \( u = (\dot{x}, \dot{y}) \); the spiky ball has a mass \( m_b = 1 \text{kg} \), the rubber band is always straight between Papi and the spiky ball. Let the length of the rubber band be, \( L \), the distance between Papi and the spiky ball. The rubber band force is

\[
F(L) = k \cdot L
\]

where \( k = 1 \text{ N/m} \). In addition, the spiky ball also experiences a damping force given by:

\[
F_{\text{damp}} = -b v_b
\]

where \( b = 2 \text{N/(m/s)} \) is the damping coefficient, \( v_b = [\dot{x}_b; \dot{y}_b] \) is the velocity of the spiky ball.

a) Using the global coordinates \( (x_b, y_b) \) of the spiky ball and \( (x, y) \) of Papi as part of the states, specify a minimal set of state variables for this system. Derive the state dynamics and the output equations that specify the position of spiky ball. [Hint: Resolve \( F(L) \) into the Cartesian coordinates, \( F_x \) and \( F_y \) and you should see a linear relationship. If you wish, you can also use Lagrange equation.]

b) Code up your model in Matlab or Simulink, and simulate the system for several secs (as long as it is interesting!) If using Matlab, learn to use ODE23 (or ODE45). Plot the position of the spiky ball in the x-y plane in each of the follow cases

![Figure 1: Papi-rubber game](image-url)
(a) with no-inputs (i.e. Papi is stationary) and with initial conditions \((x, y) = (0, 0)\), \((x_b, y_b) = (1, 0), \ (\dot{x}_b, \dot{y}_b) = (0, -1)\).

(b) with initial conditions \((x, y) = (0.1, 0), \ (x_b, y_b) = (0.2, 0), \ (\dot{x}_b, \dot{y}_b) = (0, 0)\), and with input
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = 2\pi \begin{pmatrix} y \\ -x \end{pmatrix}
\]

2. **Linearization** Consider the system with no input,
\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -5(x_1^2 - 1)x_2 - x_1 \end{pmatrix}
\]
The nominal initial condition is \((x_1, x_2)^T = (0, 2)^T\).

(a) Simulate this system for 40s to obtain a nominal state trajectory \(\bar{x}(t)\). Plot \(\bar{x}_2(t)\) against \(\bar{x}_1(t)\)

(b) Write down the linearized states space system about \(\bar{x}(t)\), i.e. \(\dot{x} = A(t)\dot{x}\). Okay to leave \(\bar{x}(t)\) in the expressions for \(A(t)\).

(c) Compute the transition matrix \(\Phi(t, 0)\) and plot its elements (as function of time).

(d) Let the initial state of the nonlinear system be \(x(t=0) = (0.2, 1.8)^T\). Estimate the response of the nonlinear system using the transition matrix obtained previously for the linearized system. Compare (via a time plot) the actual nonlinear system response and the estimated response (as a function of time).

3. **Transition matrix for a time varying system** Consider a periodic system:
\[
\dot{x} = \begin{pmatrix} 0 \\ -(1.5 - \sin(2\pi t)) & 1 \\ -0.3 \\ -1 \end{pmatrix} x; \quad x \in \mathbb{R}^2
\]
Outline the procedure for computing its transition matrix. Then, numerically compute \(\Phi(2, 0)\) and \(\Phi(4, 0)\) [Hint: Use a first order integration - e.g. to compute \(\dot{x} = f(x)\) use \(x(t+h) \approx x(t) + h \cdot f(x(t))\) for \(h\) sufficiently small; or better yet, learn to use ODE23 / ODE45 in Matlab.]. Determine what \(x(t=0)\) has to be in order to have
\[
\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} x(2) - x(4) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
Check your answer by simulating it.

4. For the matrix \(A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}\), find its transition matrix \(\Phi(t, 0) = \exp(At)\) by

(a) Modal decomposition (eigenvector / eigenvalue) method. You may use Matlab.
(b) Laplace transform.

5. Find the transition matrix for the matrix,
\[
A(t) = \begin{pmatrix} \sin(t) - \cos(t) & \sin(t) + \cos(t) \\ \sin(t) + \cos(t) & \sin(t) - \cos(t) \end{pmatrix}
\]
[Hint: Find it’s eigenvectors.]
6. In this problem, you will experiment with the geometric meaning of the transition matrix as it transforms a set of initial states to their final states.

For each of the matrix below given in the eigen-decomposition,

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\]

\[
A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\]

\[
A_3 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}
\]

determine the transition matrices \(\exp(A_i t)\), at \(t = 0.1, 1, 5\). You can use the Matlab command \(\text{expm}\) if you wish.

Compute and sketch (plot) where a square with vertices \((0, 0), (0, 1), (1, 1), (1, 0)\) is mapped to by the transition matrices at \(t = 0.1, 1, 5\).

Compare the volumes (i.e the areas in this case) of images of the square with the determinant of the transition matrices at various times.

What can you say about the shape of the images for large \(t\)? (Hint: with respect to the eigen vectors.)

7. (Peano-Baker formula) For \(A(t) = \alpha(t)M_1 + \beta(t)M_2\) where \(M_1\) and \(M_2\) are constant matrices such that \(M_1 M_2 = M_2 M_1\) and \(\alpha(t), \beta(t)\) are scalar functions, show, using the Peano-Baker formula that the transition matrix is exactly the matrix exponential:

\[
\Phi(t, 0) = \exp(\int_0^t A(\tau) d\tau)
\]

8. (Periodic system - Floquet theorem) Consider a periodic system \(\dot{x} = A(t)x\) where \(A(t) = A(t + T)\). In this problem you will find its transition matrix.

   (a) Show that \(\Phi(t, 0) = \Phi(t + T, T)\) for all \(t\).
   (b) Show that \(\Phi(kT, 0) = \Phi(T, 0)^k\).
   (c) Write down the transition matrix \(\Phi(t_1, t_0)\) in terms of \(\Phi(T, 0), \Phi(\tau_0, 0), \Phi(\tau_1, 0)\) and \(k\) where \(0 \leq \tau_0, \tau_1 < T\), and \((t_1 - t_0 - T)/T < k \leq (t_1 - t_0)/T\).