EXAMPLE:

Air flows in a 5-foot diameter pipe whose relative roughness \( R/D = 0.0001 \). At station 1, \( p_1 = 50 \) psia and \( V_1 = 50 \) ft/sec. The temperature \( T = 80^\circ F \) is constant. Station 2 is 5 miles downstream of station 1.

Find: (a) \( p_2 \), (b) \( M_2 \), (c) \( \Delta Q \) between stations 1 and 2.

Game plan: Use given data at state 1 to get \( (4f \frac{L_{max}}{D})_1 \). Move from state 1 to state 2 by means of \( 4f \frac{L}{D} \), where \( L \) is the distance between states 1 and 2. Compute the requested properties at 2.

State 1: \[ M_1 = \frac{50}{49.02 \sqrt{80 + 459.7}} = 0.0439 \]

\[ K M_1^2 = 0.00270 \]

\( (4f \frac{L_{max}}{D})_1 = \frac{1}{K M_1^2} - 1 + \ln K M_1^2 = 363.46 \)

State 1 to state 2:

\[ (4f \frac{L_{max}}{D})_2 = (4f \frac{L_{max}}{D})_1 - 4f \frac{L}{D} \]

\[ = 363.46 - \frac{4f (5280)}{8} \]
Isothermal, frictional flow

The value of \(4f\) is needed.

At \(T = 80^\circ F\), \(\mu = 1.24 \times 10^{-5} \frac{\text{lbm}}{\text{ft-sec}}\)

Next, to find \(\text{Re}\), \(m\) is needed.

\[
\dot{m} = \rho_i A_i V_i = \frac{p_i}{R T_i} A_i V_i
\]

\[
m = \frac{(50)(144)}{(53.35)(539.7)} (\frac{\pi}{4})(5^2) 50 = 245.44 \text{ lbm/sec}
\]

\[
\text{Re} = \frac{4m}{\mu \pi D} = \frac{(4)(245.44)}{(1.24)10^{-5}} \pi (5) = 5.04 \times 10^6
\]

From the chart on page 119 with \(k/D = 0.0001\), \(4f = 0.0123\).

Therefore,

\[
(4f \frac{2.5}{D})^2 = 363.46 - (0.0123)(5280)
\]

\[
= 298.52
\]

\[
= \frac{1}{K M_2^2} - 1 + \ln K M_2^2 \quad \rightarrow K M_2^2 = 0.003276
\]

\[
M_2 = \left[ \frac{(0.003276)}{1.4} \right]^{1/2} = 0.04837
\]

\[
\frac{P_1}{P'} = \frac{1}{M_2 \sqrt{k}} \quad \text{and} \quad \frac{P_2}{P'} = \frac{1}{M_2 \sqrt{k}}
\]

\[
\frac{P_2}{P_1} = \frac{M_1}{M_2} \quad \text{and} \quad P_2 = \frac{0.0439}{0.04837} 50 = 45.38 \text{ psi}
\]
Isothermal, frictional flow

To determine $Q$, use page 149

$$Q = c_p \left( \frac{k-1}{2} \right) T (M_2^2 - M_1^2)$$

$+ 0.24 \text{ Btu/lbm}^{-\circ} \text{F}$ for air

$Q = (0.24) \left( \frac{1.4 - 1}{2} \right) (539.7) (0.04837^2 - 0.0439^2)$

$Q = 0.0107 \text{ Btu/lbm}$

$mQ = (245.44)(0.0107) = 2.626 \text{ Btu/sec}$

$= 9,454.3 \text{ Btu/hr} = 2,770 \text{ W}$

$$\frac{V_2}{V_1} = M \sqrt{\frac{R}{T}}$$

$$\frac{P_2}{P_1} = \frac{1}{M \sqrt{\frac{T}{R}}}$$

$p = pR/T$ at any $x$

$p_2 = pR/T$ at the end

$$\frac{p}{p_1} = \frac{p}{p_1}$$
Isothermal, frictional flow

Problem
Airflow, \( D = 5 \text{ feet} \), \( X/D = 0.0001 \)
\( V_1 = 50 \text{ ft/sec} \), \( p_1 = 50 \text{ psia} \), \( T = 80^\circ \text{F} \), constant
\( p_2 = 41.6 \text{ psia} \)
Find: \( f \) between 1 and 2
\( M_2 \)
\( Qm \) between 1 and 2

EXAMPLE:
\( \dot{m} = 110 \text{ lb}_m/\text{sec} \)
\( p_1 = 90 \text{ psia} \)
\( p_2 = 20 \text{ psia} \)
Methane gas
\( k = 1.3 \), \( R = \frac{154.5}{16.04} = 96.32 \frac{\text{ft} \cdot \text{lbf}}{\text{lb}_m \cdot \text{°R}} \)
\( \dot{m} = 7.5 \times 10^{-6} \text{ lb}_m/\text{ft} \cdot \text{sec} \)
\( \text{Re} = \frac{4 \dot{m}}{\pi \rho D} = \frac{4 \times 110}{7.5 \times 10^{-6} \times \pi \times 5} = 3.73 \times 10^6 \)
\( f = 0.0124 \)
\( p_1 = \frac{p_1}{RT} \), \( V_1 = \frac{\dot{m}}{\rho \pi D^2} \), \( M_1 = \frac{V_1}{\sqrt{\frac{KRT}{63.50/\text{°F}}}} \)
\( p_2 = \frac{p_2}{RT} \), \( V_2 = \frac{\dot{m}}{\rho_2 \pi D^2} \), \( M_2 = \frac{V_2}{63.50/\text{°F}} \)

\( M_1 \rightarrow 4f \frac{\dot{d}_{max}}{D} \), \( M_2 \rightarrow 4f \frac{\dot{d}_{max}}{D} \)

\( f = \frac{D}{4f} \left\{ \left( 4f \frac{\dot{d}_{max}}{D} \right)_{M_1} - \left( 4f \frac{\dot{d}_{max}}{D} \right)_{M_2} \right\} \), etc.
HEAT TRANSFER IN A CONSTATE AREA WITH NEGLIGIBLE FRICTION

- Heat transfer at the duct wall
- Combustion in the flowing gas in the presence of considerable excess air (to cause properties to be nearly constant).

**Energy Equation**

\[
Q_1 = \left( h + \frac{V_1^2}{2} \right)_2 - \left( h + \frac{V_1^2}{2} \right)_1
\]

\[
Q = c_p \left( T + \frac{V_1^2}{2c_p} \right)_2 - c_p \left( T + \frac{V_1^2}{2c_p} \right)_1,
\]

\[
Q = c_p (T_{o2} - T_{o1}), \quad T_{o2} = T_{o1} + \frac{Q}{c_p}
\]

\[
\frac{T_{o2}}{T_{o1}} = 1 + \frac{Q}{c_p T_{o1}}
\]

\[
\frac{T_{o2}}{T_{o1}} = \frac{T_2 \left( 1 + \frac{k-1}{2} \frac{M_2^2}{M_1^2} \right)}{T_1 \left( 1 + \frac{k-1}{2} \frac{M_2^2}{M_1^2} \right)}
\]

**Momentum Equation (negligible friction)**

\[
m (V_2 - V_1) = p_1 A - p_2 A
\]

\[
p_2 A V_2^2 - p_1 A V_1^2 = (p_1 - p_2) A
\]

Substitute \( \rho V^2 = k p M^2 \) and get
Heat transfer; negligible friction

\[ \frac{P_2}{P_1} = \frac{1 + k M_1^2}{1 + k M_2^2} \]  \hspace{1cm} (2)

**Mass Conservation**

\[ \rho_1 V_1 = \rho_2 V_2 \quad \text{or} \quad \frac{V_2}{V_1} = \frac{P_1}{P_2} \]  \hspace{1cm} (3)

**State**

\[ \frac{P_2}{P_1} = \frac{P_2}{P_1} \frac{T_2}{T_1} \]  \hspace{1cm} (4)

**Mach Number**

\[ \frac{M_2}{M_1} = \frac{V_2}{V_1} \sqrt{\frac{T_1}{T_2}} \]  \hspace{1cm} (5)

**Relationship Between \( T_0 \) and \( M \)**

Goal is to eliminate \( T_2/T_1 \) from equation (1). Use (3) and (5) to eliminate \( V_2/V_1 \), and then eliminate \( P_2/P_1 \), giving

\[ \frac{P_2}{P_1} = \frac{M_1}{M_2} \sqrt{\frac{T_2}{T_1}} \]

Combine this with (2) and get

\[ \frac{T_2}{T_1} = \left( \frac{M_2}{M_1} \right)^2 \left( \frac{1 + k M_1^2}{1 + k M_2^2} \right)^2 \]

and, after substitution in (1),

\[ \frac{T_{02}}{T_{01}} = \left( \frac{M_2}{M_1} \right)^2 \left( \frac{1 + k M_1^2}{1 + k M_2^2} \right)^2 \left( \frac{1 + \frac{k-1}{2} M_2^2}{1 + \frac{k-1}{2} M_1^2} \right) \]

\( T_0 \) drives \( T_0 \) changes, and \( T_0 \) drives changes in \( M \).
Heat transfer, negligible friction

For a tabular presentation, let

\[ T_{01} \rightarrow T_0, \quad T_{02} \rightarrow T_0^\prime, \quad M_1 \rightarrow M, \quad M_2 = 1 \]

\[ \frac{T_0}{T_0^\prime} = 2(k+1)M^2 \cdot \frac{1 + \frac{k-1}{2}M^2}{(1 + kM^2)^2} \]

Note that heating \((Q > 0)\) increases \(T_0\) (page 155) and that cooling \((Q < 0)\) decreases \(T_0\). Therefore, as shown in the graph:

- Heating drives \(M\) toward 1 both for \(M < 1\) and \(M > 1\).
- Cooling drives \(M\) away from 1 both for \(M < 1\) and \(M > 1\).

For every \(M\), the difference between \(\frac{T_0}{T_0^\prime}\) and 1 is the extent of the maximum possible heating.
This figure shows that there is a greater potential for heating on the subsonic side than on the supersonic side. This suggests that a shock situated in a heated duct—transforming an $M>1$ flow into an $M<1$ flow—would act to increase the amount of heat that could be added to the flow. To examine this issue, apply the $T_0/T_0''$ equation (top of page 157) at both the x and y sides of a shock in a heated duct.

\[
\left(\frac{T_0}{T_0''}\right)_x = 2(k+1)M_x^2 \frac{1 + \frac{k-1}{2}M_x^2}{(1 + KM_x^2)^2}; \quad \left(\frac{T_0}{T_0''}\right)_y = 2(k+1)M_y^2 \frac{1 + \frac{k-1}{2}M_y^2}{(1 + KM_y^2)^2}
\]

But, from page 76 of SHOCKS

\[
\frac{M_x \sqrt{1 + \frac{k-1}{2}M_x^2}}{1 + KM_x^2} = \frac{M_y \sqrt{1 + \frac{k-1}{2}M_y^2}}{1 + KM_y^2}
\]

or

\[
\frac{M_x^2 (1 + \frac{k-1}{2}M_x^2)}{(1 + KM_x^2)^2} = \frac{M_y^2 (1 + \frac{k-1}{2}M_y^2)}{(1 + KM_y^2)^2}
\]

which leads to

\[
\left(\frac{T_0}{T_0''}\right)_x = \left(\frac{T_0}{T_0''}\right)_y
\]

The graphical interpretation of this result is...
Heat transfer, negligible friction

The figure shows that the difference between $T_0/T_0''$ and 1.0 does not change across a shock.

Suppose a shock occurred in the diverging part of a nozzle which feeds a heated duct.

The shock at the duct inlet has no effect on $T_0/T_0''$, but a shock in the nozzle leads to a lowering of $T_0/T_0''$ at the duct inlet.
Heat transfer, negligible friction

Relating $p, T, V, \ldots$ to $M$

Apply the $p'/p$ equation from the top of page 156 to two states: $(p, M)$ and $(p'', M=1)$ and get

\[ \frac{p''}{p'} = \frac{1+k}{1+kM^2} \]

and similarly for the other quantities.

\[ \frac{T''}{T'} = M^2 \left( \frac{1+k}{1+kM^2} \right) \]

\[ \frac{V''}{V'} = \frac{\rho''}{\rho} = \frac{(k+1)M^2}{1+kM^2} \]

\[ \frac{p''}{p'} = \frac{1+k}{1+kM^2} \left[ \frac{2(1+kM^2)}{1+k} \right]^{k-1} \]

Heating causes $p_0$ to decrease.

Table BS5 for $k = 1.4$:

\[ \frac{p''}{p} \]

\[ \frac{M}{T_0/T_0''} \quad \frac{T/T''} \quad \frac{p/p''} \quad \frac{p_0/p_0''} = \frac{V''}{V'} \]
Heat transfer, negligible friction

Constancy of $p''$

\[ p'' \text{AM equation: } \dot{m} = p'' \text{AM} \sqrt{\frac{k}{RT_0}} \sqrt{1 + \frac{k-1}{2} M^2} \]  
(p. 27)

Introduce $p/p''$ from page 160 and $T_0'/T_0''$ from page 157 and get

\[ \dot{m} = p'' A \sqrt{\frac{k(k+1)}{2RT_0''}} \]

\[ \alpha = \frac{\dot{m}}{A} \sqrt{\frac{2RT_0''}{k(k+1)}} \]

Since $\dot{m}$, $A$, and $T_0''$ are constant along a heated or cooled duct, so is $p''$. In particular, $p''$ is constant across a shock.

**EXAMPLE**

\[ P_1 = 63 \text{ psia, } V_1 = 250 \text{ ft/sec} \]

\[ T_1 = 537.4 \text{ °R} \]

\[ Q = 170.69 \text{ Btu/lbm} \]

\[ P_2 = ?, \quad M_2 = ? \]

\[ M_1 = V_1 / 49.02 \sqrt{T_1} = \frac{250}{49.02 \sqrt{537.4}} = 0.22 \]

\[ M_1 = 0.22 \rightarrow B_2 \rightarrow \frac{T_1}{T_0_1} = 0.99041 \rightarrow T_0_1 = 542.60 \text{ °R} \]

\[ T_1 = 537.4 \text{ °R} \]

\[ M_1 = 0.22 \rightarrow 3.5 \rightarrow T_0_1 / T_0'' = 0.20574 \rightarrow T_0'' = 2637.3 \text{ °R} \]

\[ P_1 / p'' = 2.24770 \rightarrow p'' = 28.03 \text{ psi} \]

\[ R = 63 \]

\[ T_{02} = T_{01} + \frac{Q}{c_p} = 542.60 + \frac{170.69}{0.29} = 1253.81 \text{ °R} \]
Heat transfer, negligible friction

\[ \frac{T_{02}}{T_0''} = \frac{1253.81}{2637.31} = 0.4754 \]

\[ \frac{P_2}{P_0''} = 2.01400 \rightarrow P_2 = 56.45 \quad \text{psia} \]

\[ \mu_2 = 0.37 \]

**EXAMPLE:**

\[ M_1 = 2, \quad T_2 = -160^\circ F \]

\[ p_1 = 10 \text{ psia} \quad \text{If} \quad M_2 = 1 \quad \text{what is} \quad T_{02} \? \quad \text{Also,} \quad mQ = ? \]

\[ T_{01} = T_i \left(1 + \frac{k-1}{2} M_1^2\right) = 299.67 \left(1 + 0.2 	imes 4\right) = 539.41^\circ R \]

or, from B5

\[ M_1 = 2 \quad \frac{T_i}{T_{01}} = 0.5555 \quad \begin{align*} T_i & = 299.67 \quad \checkmark \\ T_{01} & = 539.41^\circ R \end{align*} \]

\[ M_1 = 2 \quad \frac{B5}{T_{01}} = 0.7934 \quad \begin{align*} T_1 & = 479.87^\circ R \\ T_{01} & = 679.87^\circ R \end{align*} \]

To calculate \( mQ \), we must first find \( m \).

\[ m = \rho A V_i \quad \Rightarrow \quad V_i = 49.02 \frac{M_i}{\sqrt{T_i}} \]

\[ V_i = (49.02) \frac{(2)}{299.67} = 1697.17 \text{ ft/sec} \]

\[ p_1 = \frac{p_i}{RT_i} = \frac{(10)(144)}{(53.75)(299.67)} = 0.09007 \text{ lbm} \]

\[ m = (0.09007) \left(\frac{\pi}{4} \frac{2^2}{144}\right) (1697.17) = 3.335 \text{ lbm/sec} \]

\[ Q = c_p \left(T_{02} - T_{01}\right) = c_p \left(T_{0''} - T_{01}\right) \]

\[ mQ = (3.335)(0.24)(679.87 - 539.41) = 112.42 \text{ Btu/sec} \]

**EXAMPLE:**

Shock location problem (heating, negligible friction)
Heat transfer, negligible friction

\[ P_0 = 100 \text{ psi} \]
\[ T_0 = 540 \text{ °R} \]
\[ A_0 = 11 \text{ in}^2 \]
\[ A_1 = 1.518 \text{ in}^2 \]
\[ Q = 20.01 \text{ Btu/ft} \]

a) Is there a shock?

b) If there is a shock find its location.

Assume \( M = 1 \) at nozzle throat.

- Follow subsonic path.

\[ \frac{A_1}{A_0} = \frac{1.518}{11} = 0.138 \quad \text{B.2} \]

\[ M_1 = 0.48 \]

\[ \frac{P_1}{P_0} = 0.9541 \quad \text{B.2} \]

\[ P_1 = 85.41 \]

\[ \frac{P_1}{P''} = \frac{1.817466}{85.41} \quad \text{B.5} \]

\[ P'' = 47.07 \]

\[ T_1'' = 0.66139 \]

\[ T_0'' = 816.46 \]

\[ T_{02} = T_{01} + \frac{Q}{\dot{q}} = 540 + 20.01/(0.24) = 623.38 \]

\[ \frac{T_{02}}{T_0''} = \frac{623.38}{816.41} = 0.7635 \quad \text{B.5} \]

\[ \frac{P''}{P_0} = 1.675 \quad \text{B.2} \]

\[ \frac{P''}{P_2} = 47.07 \]

\[ P_2 = 78.84 \]
Heat transfer, negligible friction

\[ A_1 / A_\star = 1.380 \quad (B.2) \quad M_1 = 1.743 \]

\[ \frac{P_1}{P_\circ} = \frac{1.997}{1} \quad \Rightarrow \quad P_1 = 18.97 \]

\[ M_{1x} = 1.743 \quad (B.3) \quad \frac{P_\circ}{P_\circ} = 3.078 \quad \Rightarrow \quad P_\circ = 64.68 \quad \text{at} \quad 10 \]

\[ M_{1y} = 0.6297 \quad (D.5) \quad \frac{P_\circ}{P_\circ} = 1.54275 \quad \Rightarrow \quad P_\circ = 41.53 \]

\[ T_{02} = T_{01} + \frac{Q}{c_p} = 625.38 \text{ as before} \]

\[ \frac{T_{02}}{T_\circ} = \frac{625.38}{635.52} = 0.9805 \quad (B.5) \quad \frac{P_\circ}{P_\circ} = 1.19314 \quad \Rightarrow \quad P_\circ = 49.55 \]

(c)

There is a shock in the nozzle, as shown in the diagram.

\[ m = \frac{0.522 P_\circ A_\star}{T_{01}} = \frac{P_\circ A_\star}{T_{01}} \frac{B_5}{R T_{01}^2} \sqrt{\frac{1 - k}{k}} \sqrt{1 + \frac{k - 1}{2} M_*^2} \]

\[ \frac{(0.532)(100)(1.1)}{540} = (51)(1.518) M_* \frac{\sqrt{(1.4)(32.15)}}{(53.35/622.33)} \sqrt{1 + 0.2 M_*^2} \]

\[ M_* = 0.629 \quad \frac{B_5}{T_{02}} = 0.9745 \quad \Rightarrow \quad T_{02} = 623.38 \]

\[ \frac{P_\circ}{P_\circ} = 1.223 \quad \Rightarrow \quad P_\circ = 51 \]

\[ M_2 = 0.540 \quad \frac{B_5}{T_{01}} = 0.8442 \quad \Rightarrow \quad M_2 = 0.624 \quad \frac{B_5}{A_\star} = 1.161 \quad \Rightarrow \quad A_\star = 1.518 \quad \text{and} \quad A_y = 1.328 \]
Heat transfer, negligible friction

\[ \frac{A_x}{A_y} = \frac{1.1}{1.3075} = 0.8413 \quad \text{B3} \quad M_x = 1.734 \]

(b) \( M_x = 1.734 \quad \text{B2} \quad \frac{A_x}{A_x^*} = 1.370 \quad \rightarrow A_x = 1.507 \text{ in}^2 \)
\( A_x^* = 1.1 \)
Temperature Recovery - The Adiabatic Wall Temperature

In an adiabatic flow, the temperature decreases as the velocity increases as given by
\[ T = T_0 - \frac{V^2}{2C_p} \]

This temperature decrease may be regarded as a loss of temperature. If the flow is decelerated to zero, will the temperature loss be recovered? The answer to this question depends on the manner in which the flow is slowed down.

Consider flow along a wall. In reality, the velocity is zero at the wall and increases rapidly with increasing distance from the wall.

In the near-wall region, there is a high rate of dissipation of mechanical energy into heat. In general, some of this heat will pass into the wall and some will diffuse into the free stream. The presence of the heat-generating...
Temperature recovery

Near-wall layer causes the wall temperature to be elevated. The maximum amount of elevation will occur when no heat passes into the wall (adiabatic wall). This is because all the heat must diffuse into the free stream, and because heat flows from high to low temperature, the wall temperature must be sufficiently elevated above the free stream temperature. The temperature profile in the fluid in the presence of a heat-generating near-wall layer and an adiabatic wall is

The temperature of the adiabatic wall is called $T_w$. It is also called $T_r$ - the recovery temperature. Since the velocity at the wall is zero, it might be expected that some of the temperature lost when the fluid was accelerated adiabatically from $V=0$ to $V=V$ would be recovered.
Temperature recovery

To quantify the extent of the temperature recovery, a recovery factor $R$ is defined as:

$$ R = \frac{T_{aw} - T}{T_0 - T} $$

To implement this equation, it may be recalled that:

$$ T_0 = T + \frac{V^2}{\nu c_p} $$

or

$$ T_0 = T \left(1 + \frac{k-1}{2} M^2\right) $$

so that

$$ T_{aw} = T + R \frac{V^2}{\nu c_p} $$

or

$$ T_{aw} = T \left[1 + R \frac{k-1}{2} M^2\right] $$

By ratioing $T_{aw}$ and $T_0$, there follows:

$$ \frac{T_{aw}}{T_0} = \frac{1 + R \frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2} $$

For turbulent flow parallel to a surface, $R = Pr^{\frac{1}{3}}$. For air, $Pr \approx 0.7$, so that $R \approx 0.888$. For a cylinder in crossflow (for example, a thermocouple or a thermometer), $R \approx 0.6$. 
EXAMPLE: Adiabatic-walled circular pipe with an inner diameter D = 2 inches.

At station 1, \( p_1 = 69.27 \text{ psia}, M_1 = 0.46 \),

\[ (T_{aw})_1 = 531.62^\circ R \] the recovery factor

\[ \Theta = 0.888. \]

\[ (T_{aw})_2 = ? \] Station 2 is 22.27 feet downstream of station 1.

At station 1:

\[ (T_{aw})_1 = T_0 \left\{ \frac{1 + \Theta \frac{k-1}{2} M_1^2}{1 + \frac{k-1}{2} M_1^2} \right\} \]

With \( (T_{aw})_1 = 531.62^\circ R \), \( M_1 = 0.46 \), \( k = 1.4 \),

the solution for \( T_0 \) gives \( T_0 = 534.04^\circ R \).

Note that \( T_0 \) is constant since duct is adiabatic. Then,

\[ T_1 = \frac{(T_{aw})_1}{1 + \Theta \frac{k-1}{2} M_1^2} = 512.365^\circ R \]

For now, assume

\[ T = T_1 = 512.365^\circ R = 52.695^\circ F \]

Then, \( \mu = 1.1925 \times 10^{-5} \text{ lb}_m/\text{ft} \cdot \text{sec} \)

Also,

\[ \rho_1 = \frac{p_1}{RT_1} = 0.3333 \text{ lb}_m/\text{ft}^3 \]

\[ V_1 = M_1 \left( 49.02 \sqrt{T_1} \right) = 510.41 \text{ ft/sec} \]

\[ m = \rho_1 AV_1 = 3.708 \text{ lb}_m/\text{sec} \]

\[ Re = \frac{4 \bar{m}}{\mu \pi D^2} = 2.375 \times 10^6 \quad \Rightarrow \quad 4 \bar{F} = 0.01 \quad \text{(smooth tube)} \]
Use BY, with $M_1 = 0.46$
\[
(4\bar{f}L_{\text{max}}^2/D) = 1.45091
\]
Between 1 and 2, \(L/D = 22.27/(0.6) = 133.6\)
so that \(4\bar{f}(L/D) = (0.01)(133.6) = 1.336\)
Then \(4\bar{f}L_{\text{max}}^2/D) = 1.45091 - 1.336 = 0.1149\)

By

\[
M_2 = 0.76 \quad \frac{B^2}{T_0} = 0.89149 \quad \Rightarrow T_2 = 478.73 \quad M_2 = 0.76
\]
\[
T_0 = 534.04
\]

\[
(Taw)_2 = T_2 \left[ 1 + O \frac{K}{2} M_2^2 \right] = (478.73) \left[ 1 + (0.891)(0.2)(0.76)^2 \right]
\]

\[
(Taw)_2 = 527.85
\]

Now, it is necessary to check the value of \(\mu\), which was tentatively based on \(T_1\).
Compute \(\frac{1}{T} = \frac{T_1 + T_2}{2} = 495.55\)

Find the new \(\mu\), etc.

Let \(q^\prime\) be the heat flow per unit time and unit area from the wall to the fluid.
If the wall temperature \(T_w = T_{aw}\), then, by definition, \(q^\prime = 0\). If \(T_w > T_{aw}\), \(q^\prime\) is positive, i.e., heat flows from the wall to the fluid. If \(T_w < T_{aw}\), \(q^\prime\) is negative and
Temperature recovery

heat flows from the fluid to the wall. This suggests that

\[ q'' = h (T_w - T_a) \]

where \( h \) is the convective heat transfer coefficient. If \( T_a \) is introduced

\[ q'' = h \left[ T_w - T \left( 1 + \varepsilon \frac{E}{h} \frac{h}{2M^2} \right) \right] \]

For low speed flow, \( M \to 0 \),

\[ q'' = h (T_w - T) \]

which is the conventional definition.
The processes to be considered are combined area change, friction, and heat transfer.

The conservation equations relevant to these processes were derived previously and are repeated here for convenience.

mass: \[ \frac{dp}{p} + \frac{dA}{A} + \frac{dV}{V} = 0 \]

momentum: \[ \frac{dp}{p} + \frac{kM^2}{2} \frac{dV}{V} + \frac{kM^2}{2} \frac{4fdx}{D} = 0 \]

energy: \[ c_p dT + d\left(\frac{V^2}{2}\right) = d\mathcal{Q} = c_p dT_0 \]

state: \[ \frac{dp}{\rho} = \frac{dp}{\rho} + \frac{dT}{T} \]

Mach no: \[ \frac{dM}{M} = \frac{dV}{V} - \frac{1}{2} \frac{dT}{T} \]

These equations can be manipulated to relate changes in quantities such as \( M, V, \rho, \) etc. to the cause of change such as \( dA/A, dT_0/T_0, \) and \( 4fdx/D. \)

The end result of these manipulations is presented in the table on page 173, and the equation for \( dM^2/M^2 \) appearing at the bottom of the table illustrates its use. In reality, the only differential equation
### Influence Coefficients for Constant Specific Heat and Molecular Weight

<table>
<thead>
<tr>
<th></th>
<th>( \frac{dA}{A} )</th>
<th>( \frac{dT_v}{T_v} )</th>
<th>( \frac{4f}{D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dM^3}{M^3} )</td>
<td>(-\frac{2\left(1 + \frac{k-1}{2}M^3\right)}{1-M^3})</td>
<td>(\frac{(1 + kM^3)(1 + \frac{k-1}{2}M^3)}{1-M^3})</td>
<td>(\frac{\alpha M^3 \left(1 + \frac{k-1}{2}M^3\right)}{1-M^3})</td>
</tr>
<tr>
<td>( \frac{dV}{V} )</td>
<td>(-\frac{1}{1-M^3})</td>
<td>(\frac{1 + \frac{k-1}{2}M^3}{1-M^3})</td>
<td>(\frac{\alpha M^3}{2(1-M^3)})</td>
</tr>
<tr>
<td>( \frac{d\bar{e}}{\bar{e}} )</td>
<td>(\frac{k-1}{2}\frac{M^3}{1-M^3})</td>
<td>(\frac{1 - kM^3}{2} \left(1 + \frac{k-1}{2}M^3\right))</td>
<td>(-\frac{\alpha (k-1)M^3}{4(1-M^3)})</td>
</tr>
<tr>
<td>( \frac{dT}{T} )</td>
<td>(\frac{(k-1)M^3}{1-M^3})</td>
<td>(\frac{(1 - kM^3)(1 + \frac{k-1}{2}M^3)}{1-M^3})</td>
<td>(-\frac{\alpha (k-1)M^3}{2(1-M^3)})</td>
</tr>
<tr>
<td>( \frac{\delta e}{\delta e} )</td>
<td>(\frac{M^3}{1-M^3})</td>
<td>(-\frac{1 + \frac{k-1}{2}M^3}{1-M^3})</td>
<td>(-\frac{\alpha M^3}{2(1-M^3)})</td>
</tr>
<tr>
<td>( \frac{d\rho}{\rho} )</td>
<td>(\frac{\alpha M^3}{1-M^3})</td>
<td>(-\frac{\alpha M^3 \left(1 + \frac{k-1}{2}M^3\right)}{1-M^3})</td>
<td>(-\frac{\alpha M^3 (1 + (k-1)M^3)}{2(1-M^3)})</td>
</tr>
<tr>
<td>( \frac{d\rho}{\rho_0} )</td>
<td>(0)</td>
<td>(-\frac{\alpha M^3}{2})</td>
<td>(-\frac{\alpha M^3}{2})</td>
</tr>
<tr>
<td>( \frac{dP}{P} )</td>
<td>(\frac{1}{1 + \alpha M^3})</td>
<td>(0)</td>
<td>(-\frac{\alpha M^3}{2})</td>
</tr>
<tr>
<td>( \frac{d\bar{e}}{\bar{e}_p} )</td>
<td>(0)</td>
<td>(1 + \frac{k-1}{2}M^3)</td>
<td>(-\frac{\alpha M^3 (k-1)M^3}{2})</td>
</tr>
</tbody>
</table>

**Note:** Each influence coefficient represents the partial derivative of the variable in the left-hand column with respect to the variable in the top row. For example,

\[
\frac{dM^3}{M^3} = -\frac{2\left(1 + \frac{k-1}{2}M^3\right)}{1-M^3} \frac{dA}{A} + \frac{(1 + kM^3)(1 + \frac{k-1}{2}M^3)}{1-M^3} \frac{dT_v}{T_v} + \frac{\alpha M^3 \left(1 + \frac{k-1}{2}M^3\right)}{1-M^3} \left(\frac{4f}{D}\right)
\]
which need be solved is the $\frac{dM^2}{M^2}$ differential equation, since changes in $P, V, T$, etc. can be related to changes in $M$ by algebraic equations.

Combined Friction and Area Change

(no heat transfer)

The differential equation for $M$ follows by setting $dT_x = 0$ for the equation at the bottom of page 173.

$$\frac{dM^2}{M^2} = 2\frac{dM}{M} = \frac{1 + \frac{k-1}{2} M^2}{1 - M^2} \left[-2 \frac{dA}{A} + kM^2g \frac{dx}{D}\right]$$

It is desired to find $M$ vs $x$, given $A = A(x)$ and assuming $f$ is a constant.

Since $A = A(x)$

$$\frac{dA}{A} = \frac{1}{A} \left(\frac{dA}{dx}\right) dx = G(x) \frac{dx}{dx}$$

When $dA/A$ is substituted into the $dM$ equation, there follows

$$\frac{dM}{dx} = \frac{M}{2} \frac{1 + \frac{k-1}{2} M^2}{1 - M^2} \left(-2 G + kM^2 \frac{4f}{D}\right) = F(M, x)$$

This equation will be solved subject to given information at a location $x = x_0$. In fact, everything is known at $x = x_0$; for example, $M_0, P_0$, etc., including $M_0$.  

Combined processes
friction, area change

The solution must, except for very special cases, be carried out numerically. A discussion of suitable numerical approaches will be presented shortly. For now it will be assumed that the $M$ vs. $x$ variation has been found, e.g.,

![Graph showing $M$ vs. $x$](image)

Note that the numerical solution provides results at a discrete number of points. The graph indicates points that are equidistant in $x$; however, a variable step size in $x$ can also be used.

Suppose that it is desired to find $p$, $T$, $V$, $p$, and $p_0$ at a station $x_i$ where $M$ has been determined by numerical integration. As already noted at the top of page 174, there is no need to solve differential equations for $dp/p$, $dT/T$, etc.
Combined processes
friction, area change

provided \( M \) vs. \( x \) has been determined. For determining \( P_{x_i} \), the starting point is

\[
m_{x_i} = m_{x_0}
\]

and using the pAM equation from the bottom of page 27,

\[
\left[ \text{pAM} \sqrt{\frac{K}{R T_0}} \sqrt{1 + \frac{k-1}{2} M^2} \right]_{x_i}
\]

\[
= \left[ \text{pAM} \sqrt{\frac{K}{R T_0}} \sqrt{1 + \frac{k-1}{2} M^2} \right]_{x_0}
\]

so that

\[
P_{x_i} = P_{x_0} \frac{A_{x_0}}{A_{x_i}} \frac{M_{x_0}}{M_{x_i}} \left( 1 + \frac{k-1}{2} M_{x_0}^2 \right) \left( 1 + \frac{k-1}{2} M_{x_i}^2 \right)
\]

Next, \( T_{x_i} \) is to be found. Since

\[
(T_0)_{x_i} = (T_0)_{x_0}
\]
Combined processes
friction, area change

or

\[ T \left( 1 + \frac{k-1}{2} M^2 \right) \]_{x_0}

= \left[ T \left( 1 + \frac{k-1}{2} M^2 \right) \right]_{x_0}

or

\[ T_{x_i} = T_{x_0} \frac{1 + \frac{k-1}{2} M_{x_0}^2}{1 + \frac{k-1}{2} M_{x_i}^2} \]

with \( T_{x_0}, M_{x_0}, \) and \( M_{x_i} \) known. Then,

\[ P_{x_i} = \frac{P_{x_i}}{RT_{x_i}} \]

\[ V_{x_i} = \frac{m}{P_{x_i} A_{x_i}} \]

Finally, from the definition of \( p_0 \) (page 38),

\[ (P_0)_{x_i} = P_{x_i} \left( 1 + \frac{k-1}{2} M_{x_i}^2 \right)^{\frac{k}{k-1}} \]

**Constant** \( M \) for friction and area change

Inspection of the \( dM^2/M^2 \) equation in the middle of page 174 indicates that if \( dA/A > 0 \), the effects of area change and friction are opposed. Therefore, it
Combined processes
friction, area change

appears that a particular \( A = A(x) \) could be determined so that \( M \) would be constant along the duct, i.e., \( dM = 0 \). If \( dM = 0 \) is introduced into the equation in the middle of page 174, then

\[
2 \frac{dA}{A} = k M^2 4 \pi \frac{dx}{D}
\]

Also,

\[
A = \pi D^2/4, \quad dA/A = 2dD/D
\]

so that

\[
dD = k M^2 f dx
\]

If, as a first approximation, \( f \) is assumed to be constant, then, since \( M \) is also constant,

\[
D_{x_f} = D_{x_0} + k M^2 f (x_f - x_0)
\]

This is the equation of a conical duct.

\[
\begin{array}{c}
\downarrow \\
\hline \\
\downarrow \\
\hline \\
D_{x_0} \\
\hline \\
D_{x_f}
\end{array}
\]
Combined processes
friction, area change

There are some interesting corollaries of the \( M = \text{constant} \) condition. For example, from the \( T_{x_a} \) equation on page 177

\[ T_{x_a} = T_{x_0} \text{ or } T = \text{constant} \]

Also, since \( M = \frac{V}{\sqrt{kRT}} \), and with \( M = \text{constant} \) and \( T = \text{constant} \),

\[ V = \text{constant} \]

Next, consider the more general case where the \( f = \text{constant} \) assumption is not acceptable. In general,

\[ f = f(Re), \quad Re = \frac{4m}{\mu \pi D} \text{ (round pipe)} \]

Since \( T \) is a constant when \( M = \text{constant} \) and since \( \mu = \mu(T) \) is, therefore, constant, it follows that

\[ Re \sim \frac{1}{D} = \left( \frac{4m}{\mu \pi} \right) \frac{1}{D} \text{ Const} \]

where it was noted that \( m \) is a constant. Therefore

\[ f = f\left( \frac{\text{const}}{D} \right) \]
Combined processes
numerical methods

When this information is introduced into the $dD$ equation in the middle of page 178,

$$\int_{D_{x_0}}^{D_{x_*}} \frac{dD}{f\left(\frac{const}{D}\right)} = k M^2 (x_* - x_0)$$

The left side of this equation can be integrated, either analytically or numerically, for any $f=Re$ relation. For example, if

$$4f = (1.82 \log_{10} Re - 1.64)^{-2}$$

then,

$$\frac{dD}{4f\left(\frac{const}{D}\right)} = \left[ (1.82 \log_{10} const - 1.64) - 1.82 \log_{10} \frac{D}{D} \right]^{-2} x dD$$

Numerical methods

For the just-concluded case of combined friction and area change, the differential equation for $dM/dx$ had the form (page
\[ \frac{dM}{dx} = F(M, x) \]

where
\[ F = \frac{M}{2} \left[ 1 + \frac{k-1}{2} \frac{m^2}{1-m^2} \left[ -2G + kM^2 \frac{4f}{D} \right] \right] \]

and
\[ G = G(x) = \frac{1}{A} \frac{dA}{dx} \]

with the given data \( M, p, T, V, p, p_0, T_0, \) and \( m \) at \( x = x_0 \) and with both \( G \) and \( D \) known functions of \( x \). Furthermore, with regard to the friction factor \( f \),
\[ f = f(Re) = f \left( \frac{\nu m}{\nu} \cdot \frac{1}{\mu D} \right) \]

Since
\[ \mu = \mu(T), \quad D = D(x) \]
then
\[ f = f(T, x) \]

Therefore, when the \( \mu = \mu(T) \) dependence is taken into account,
\[ \frac{dM}{dx} = \Phi(M, T, x) \]
Fortunately, the presence of \( T \) in the function \( \Phi(M, T, x) \) is not a major complication because, from page 177,

\[
T(x) = T_{x_0} \frac{1 + \frac{k-1}{2} M_{x_0}^2}{1 + \frac{k-1}{2} M(x)^2}
\]

consequently, it is sufficient to write

\[
\Phi(M, T, x) = F(M, x)
\]

or

\[
\frac{dM}{dx} = F(M, x)
\]

as before.

The simplest numerical scheme will now be described. At the initial point \( M_0, x_0 \),

\[
\left( \frac{dM}{dx} \right)_0 = F(M_0, x_0)
\]

so that the initial slope of the \( M \) vs \( x \) curve is also known. This is shown graphically in the figure. Note that the straight line representing the initial
Combined processes numerical methods

Slope has been extended to $x > x_0$. Next, consider a forward step from $x = x_0$ to $x = x_1$. If $\Delta x_{01} = (x_1 - x_0)$ is small enough, the slope $(dM/dx)_0$ may be assumed to prevail throughout the interval $x_0 \leq x \leq x_1$. It then follows that

$$M_1 = M_0 + (dM/dx)_0 \Delta x_{01}$$

Next, the slope $dM/dx$ at $x_1$ may be evaluated,

$$\left(\frac{dM}{dx}\right)_1 = F(M_1, x_1)$$
Combined processes numerical methods

and proceeding as before,

\[ M_2 = M_1 + (dM/dx)_1 \Delta x_{12} \]

and so forth as indicated in the figure.

This method may be improved by slope refinement. The improved method starts as the preceding one, but now the \( M_1 \) value computed along the \((dM/dx)_0\) slope is regarded as a tentative value and designated as \( M'_1 \)

\[ M'_1 = M_0 + (dM/dx)_0 \Delta x_{01} \]

Then, a tentative slope at \( x_i \) is computed from

\[ (dM/dx)'_i = F(M'_1, x_i) \]

and an improved slope for the range \( x_0 \leq x \leq x_i \) is obtained by averaging

\[ (dM/dx)'_0 = \frac{1}{2} \left[ (dM/dx)_0 + (dM/dx)'_i \right] \]

and, finally,

\[ M_1 = M_0 + (dM/dx)_0 \Delta x_{01} \]
Combined processes numerical methods

and so on to $x_2, x_3$, etc.

Further refinements can be made to obtain even higher accuracy. The best known step-by-step method and one that is available in computer libraries is the Runge-Kutta method. In essence, the RK method evaluates and averages four separate slopes in advancing from $x = x_i$ to $x = x_{i+1}$. The standard RK differential equation is

$$\frac{dy}{dx} = f(x, y)$$

with $y = y_i$ at $x = x_i$. The value of $y$ at $x = x_{i+1}$ is calculated from

$$y_{i+1} = y_i + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

The $K$s are evaluated in sequence

$$K_1 = f(x_i, y_i) \Delta x,$$  

based on slope at left end of interval

$$K_2 = f(x_i + \frac{\Delta x}{2}, y_i + \frac{K_1}{2}) \Delta x,$$  

based on slope at midpoint

$$K_3 = f(x_i + \frac{\Delta x}{2}, y_i + \frac{K_2}{2}) \Delta x,$$  

based on slope at midpoint

$$K_4 = f(x_i + \Delta x, y_i + K_3) \Delta x,$$  

based on slope at right end of interval
Combined processes numerical methods

\[ K_3 = f \left( x_n + \frac{\Delta x}{2}, \ y_n + \frac{K_2}{2} \right) \Delta x, \]

based on improved midpoint slope

\[ K_4 = f \left( x_n + \Delta x, \ y_n + K_3 \right) \Delta x, \]

based on slope at right end of interval

The RK code commonly available in computer libraries includes automatic error control. This feature enables the user to specify the desired accuracy of the solution, and the code selects the step size \( \Delta x \) needed to achieve the goal. The process by which this is achieved is to calculate \( y_{n+1} \) using one step \( \Delta x \), using two steps \( \Delta x/2 \), using four steps \( \Delta x/4 \), using eight steps \( \Delta x/8 \), etc. When the difference in \( y_{n+1} \)'s corresponding to two successive calculations as outlined above is less than the selected accuracy, the code moves on to calculate \( y_{n+2} \) at \( x_{n+2} \).

Although it appears that the RK method is confined to solving first-order ordinary differential equations,
Combined processes
friction, heat transfer

This is not true. Suppose that the task is to solve the second-order ordinary differential equation by the RK method.

\[
\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})
\]

First, define a function \( z = z(x, y) \) by

\[
\frac{dy}{dx} = z(x, y)
\]

so that

\[
\frac{dz}{dx} = f(x, y, z)
\]

This results in a pair of two simultaneous first-order ordinary differential equations which are solved by the RK method.

Combined friction and heat transfer

From the standpoint of heat transfer engineering, the case of combined friction and heat transfer is of greatest interest. There are a number of interesting thermal boundary conditions to be considered:

(a) Prescribed temperature at the pipe/duct wall, including uniform
combined processes  
friction, heat transfer  
wall temperature as a special case  
(b) Convective heat transfer at the  
external surface of the pipe/duct.  
(c) Prescribed heat transfer rate at the  
pipe/duct wall, including uniform  
heating as a special case  

Conservation equations  

The combined friction - heat transfer  
process is governed by the three con-  
servation equations (mass, momentum,  
and energy) and an equation of state  
(perfect gas law). When the Mach num-
ber is involved as a dependent variable,
then its equation of definition is also  
required.

The derivation of the governing equa-
tions can be short-circuited by using  
The table on page 173

$$\frac{dM^2}{M^2} = \frac{1+\frac{k-1}{2}M^2}{1-M^2} \left[ \left(1+KM^2\right) \frac{dT}{T_0} + KM^2 \frac{df}{D} \right]$$
Combined processes
friction, heat transfer

The ultimate goal is to solve this differential equation for \( M \) vs. \( x \). There are various alternative pathways that can be followed to achieve this goal depending on the thermal boundary conditions. Regardless of the boundary condition, the energy conservation condition (page 14) applies

\[
\frac{dQ}{dT} = c_p \frac{dT}{dt}
\]

The next step is to interconnect \( dQ \) with the thermal boundary conditions. The simplest interconnection is for the uniform heat flux boundary condition at the tube/duct wall. In the laboratory, uniform wall heat flux is achieved by passing electric current through a thin-walled metallic tube. The ohmic heating (\( \Delta R \) heating) is uniform provided that the wall thickness is uniform and the variation of the
Combined processes
friction, heat transfer

resistivity of the material is small. If
the external surface of the wall is
very well insulated, then all of the
uniformly dissipated heat is forced
to flow into the fluid which passes
through the tube. Let $S$ be the dis-
sipated power per unit tube length.
Then, $Sdx$ is the power (Watts) that
is transferred to the fluid in the
length $dx$. Recall that $dQ$ is the
energy received per unit mass of the fluid
in a length $dx$, and $mdQ$ is the
power received. Therefore,

$$Sdx = mdQ = mc_p dT_o$$

so that

$$dT_o = \frac{S}{mc_p} \, dx$$

The quantities $S$ and $m$ are independ-
ent of $x$, and, in a non-chemically-
reacting flow, $c_p$ is very nearly constant.
Therefore,

$$T_o(x) = \frac{S}{mc_p} x + T_o(x_0)$$

Consequently, for the uniform heat
flux boundary condition, the $dM^2$
Combined process 
friction, heat transfer

equation at the bottom of page 188 becomes

\[ \frac{dM^2}{M^2} = \frac{1 + \frac{k-1}{2}M^2}{1-M^2} \left[ \frac{1+KM^2}{S \frac{mcp}{mcp} x + T_0(x_0) \frac{mcp}{mcp} + KM^2 \frac{4f}{D}} \right] dx \]

This equation can be written in the form

\[ \frac{dM}{dx} = F(M, x) \]

with

\[ F(M, x) = \frac{M(1 + \frac{k-1}{2}M^2)}{2(1-M^2)} \left[ \frac{1+KM^2}{x + \frac{mcpT_0(x_0)}{S} + KM^2 \frac{4f}{D}} \right] \]

Also recall that if the variation of \( f \) with the Reynolds number \( Re \) is to be taken into account, then

\[ f = f(Re) = f \left( \frac{4 \frac{m}{\mu D}}{Re} \cdot \frac{1}{\mu} \right) = f \left( \frac{const \\eta}{\mu} \right) \]

Since \( \mu \) is a function of \( T \), then \( T \) must
be computed at each step of the numerical integration from (page 27)

\[ T = \frac{T_0}{1 + \frac{k-1}{2} M^2} = \frac{(S/m_c p)x + T_0(x_0)}{1 + \frac{k-1}{2} M^2} \]

The \( \text{d}M/\text{d}x \) equation of page 191, supplemented with a relation between the friction \( f \) and the temperature \( T \) and subject to the starting condition \( M = M_0 \) at \( x = x_0 \), is readily solved numerically, for example, by using the RK method. The solution generates \( M \text{ vs. } x \), and \( T_0 \text{ vs. } x \) and \( T \text{ vs. } x \) are also made available during the solution process. All other properties such as \( p, p, V, \) and \( p_0 \) can be calculated from algebraic equations following the approach described on pages 176 and 177.

When the heat transfer rate at the pipe/duct wall is prescribed, the result of greatest interest is the axial variation of the pipe wall temperature. Suppose that \( M, T_0, T \), etc. have all been determined for the problem of uniform heating at the pipe wall, and it remains
Combined processes friction, heat transfer

to find $T_w(x)$.

For this purpose, it is necessary to work
with the convective heat transfer coefficient $h$ which characterizes the heat
transfer from the inner surface of the
pipe wall to the flowing gas.

Let $q''$ be the rate of heat transfer
per unit area from the wall to the gas.
Then, for conventional low-speed gas flows,

$$q'' = h(T_w - T_b)$$

where $T_b$ is the bulk temperature of the
fluid. In this equation, $q''$, $h$, $T_w$, and $T_b$
correspond to any location $x$. As
Combined processes
friction, heat transfer

discussed on pages 170 and 171, this
equation does not apply for compressible
gas flows. Rather, as pointed out
there, the correct equation is

\[ q'' = h(T_w - T_{aw}) \]

and since (page 168)

\[ T_{aw} = T \left[ 1 + \varpi \frac{K-1}{2} M^2 \right] \]

then,

\[ T_w = \frac{q''}{h} + T \left[ 1 + \varpi \frac{K-1}{2} M^2 \right] \]

Furthermore,

\[ \frac{\text{power}}{\text{area}} \times \frac{\text{area}}{\text{length}} = \frac{\text{power}}{\text{length}} \]

area = length x
circumference

\[ A = L \times C \]
\[ A/L = C \]
Combined processes
friction, heat transfer

so that

\[ q''C = S \]

and, with \( T \) from the top of page 192,

\[
T_w = \frac{S}{hC} + \frac{[(S/mc_p)x + T_o(x_o)][1 + \frac{R}{2} \frac{k-1}{M^2}]}{1 + \frac{k-1}{2} \frac{M^2}{}}
\]

Since \( M = M(x) \) is known, then \( T_w(x) \) can readily be computed.

The next case to be considered is that in which the wall temperature \( T_w(x) \) is known but where \( q'' ( = S/C) \), \( T_o \), \( M \), \( T \), \( p \), etc., are unknown. The analysis begins with the \( dM^2 \) at the bottom of page 188. Again, the solution strategy is to eliminate \( dT_o \) from this equation. This is accomplished by taking equations from the middle of page 190, from the top of 195, and the top of page 194, i.e.,

\[ dT_o = \frac{S}{mc_p} dx, \quad q''C = S, \quad q'' = h(T_w - T_{aw}) \]

which give
Combined processes
friction, heat transfer

\[ dT_0 = \frac{hC(T_w - T_0)}{mc_p} \, dx \]

and from page 168

\[ T_{aw} = T_0 \frac{1 + \varrho \frac{K-1}{2} M^2}{1 + \frac{K-1}{2} M^2} \]

When these equations are substituted in the \( dM^2 \) equation at the bottom of page 188, there follows

\[ \frac{dM}{dx} = \frac{M \left( 1 + \frac{K-1}{2} M^2 \right)}{2 \left( 1 - M^2 \right)} \left[ \frac{1 + KM^2 \frac{hC}{T_0}}{mc_p} \left( T_w - T_0 \frac{1 + \varrho \frac{K-1}{2} M^2}{1 + \frac{K-1}{2} M^2} \right) \right. \\
\left. + \frac{km^2 \frac{4f}{D}}{D} \right] \]

This equation has the form

\[ \frac{dM}{dx} = F(M, T_0, x) \]

which is different from prior \( dM/dx \) equations because \( F \) includes two dependent variables \( M \) and \( T_0 \). Needed is an equation for \( dT_0/dx \), and this is obtained from the top
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\[
\frac{dT_o}{dx} = \frac{hC}{mc_p} \left[ T_w - T_o \frac{1 + \Omega \frac{K-1}{2} M^2}{1 + \frac{K-1}{2} M^2} \right] = G(M, T_o, x)
\]

Actually, the numerical solution requires simultaneous treatment of the two coupled, first-order, ordinary differential equations

\[
\frac{dM}{dx} = F(M, T_o, x), \quad \frac{dT_o}{dx} = G(M, T_o, x)
\]

Before proceeding with the specification of \(h, f, \) and \(R\) needed for the numerical solution, it is interesting to introduce certain modest approximations which will simplify the solution task.

The assumptions are:

(a) Recovery factor \(R = 1\)
(b) \(h = \text{constant}\)
(c) \(c_p = \text{constant}\)

Then, the \(dT_o/dx\) equation becomes
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\[ \frac{dT_o}{dx} = \frac{hC}{mc_p} (T_w - T_o) \]

or

\[ \frac{dT_o}{(T_o - T_w)} = -\frac{hC}{mc_p} \, dx \]

This equation can be integrated in closed form when the wall temperature is uniform. The integration constant

\[ \ln (T_o - T_w) = -\frac{hC}{mc_p} x + C_1 \]

is determined to satisfy

\[ T_o = T_o(x_0) \text{ at } x = x_0 \]

so that

\[ T_o(x) - T_w = (T_o(x_0) - T_w) e^{-\frac{hC}{mc_p}(x-x_0)} \]

This equation shows that \( T_o(x) \) approaches \( T_w \) as \( x \) increases.

Next, the \( dM/dx \) equation in the middle of page 196 can be reconsidered.
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After introducing $R = 1$ and the just-found expression for $T_o(x) - T_w$, there is obtained

$$\frac{dM}{dx} = \frac{M (1+\frac{k-1}{2} M^2)}{2 (1 - M^2)} \left[ (1+km^2) \frac{hc}{mc_p} \frac{1 - \frac{T_o(x)}{T_w}}{\frac{T_o(x)}{T_w}} + km^2 \frac{4f}{D} \right]$$

where

$$1 - \frac{T_o(x)}{T_w} = \frac{\left(1 - \frac{T_o(x_0)}{T_w}\right) e^{-\frac{hc}{mc_p} (x-x_0)}}{1 + \left(\frac{T_o(x_0)}{T_w} - 1\right) e^{-\frac{hc}{mc_p} (x-x_0)}}$$

thereby reducing the $dM/dx$ equation to the form

$$\frac{dM}{dx} = F(M, x)$$

which is a standard form for numerical integration.

To continue the stream of simplifying assumptions, use may be made of Reynolds analogy. This is a relation between the friction and heat
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transfer as follows

\[ \frac{f}{2} = \frac{Nu}{Re Pr} = St = \text{Stanton number} \]

where \( Nu, Re, \) and \( Pr \) are respectively
the Nusselt, Reynolds, and Prandtl numbers defined as

\[ Nu = \frac{hD}{k'}, \quad Re = \frac{um}{\mu C}, \quad Pr = \frac{c_p \mu}{k'} \]

where \( k' \) is the thermal conductivity of
the fluid. When these definitions are
introduced into Reynolds analogy,

\[ \frac{hC}{mc_p} = 2 \frac{f}{D} \]

and

\[ \frac{dM}{dx} = \frac{M(1 + \frac{k-1}{2} M^2)}{2 (1 - M^2)} \frac{4f}{D} \left[ \frac{1 + kM^2}{2} \left( 1 - \frac{T_0(x)}{T_W} \right) + kM^2 \right] \]

\[ e^{-\frac{hC}{mc_p} (x-x_0)} = e^{-\frac{2f}{D} (x-x_0)} \]

In view of all the preceding assumptions,