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\[ \dot{Q} = \dot{Q} - \dot{U} + \dot{m} \left( h_1 - \frac{V_1^2}{2} \right) - \dot{m} \left( h_2 - \frac{V_2^2}{2} \right) \]

\[ \frac{U_{max}}{U} = 2 \]

\[ \frac{\overline{U}}{\overline{V}} = 1.2 \]

\[ h = U + PV = U + RT \]

Note: For ideal gas
Introduction to compressibility

THERMODYNAMICS OF FLUID FLOW
(GAS DYNAMICS)

The focus of the course is gas flows in which there are appreciable density variations.

Causes of significant density variations:

- Pressure-related density variations
  - Large, rapid changes in cross-sectional area causing acceleration or deceleration
  - A shock (a discontinuity across which pressure, velocity, density, etc. experience very great changes)
  - Friction at the bounding wall
    - High-speed flow in short ducts
    - Low-speed flow in long ducts

- Temperature-related density variations:
  - Heat transfer at bounding walls
  - Combustion within the flow
  - Viscous dissipation within the boundary layer

Typical applications:

- Gas flows in power-producing devices
- Flow passages between turbine blades
- Combustion chambers
Introduction to compressibility

- Compressors
- Nozzles of rockets and turbojets
- Aerodynamics
- Lift and drag
- Vacuum systems
- Long gas-conveying pipelines

* Measures of compressibility

- Coefficient of thermal expansion \( \beta \):
  \[
  \beta = -\frac{1}{\rho} \left( \frac{\partial p}{\partial T} \right)_p
  \]

- Isothermal compressibility \( \alpha \):
  \[
  \alpha = \frac{1}{\rho} \left( \frac{\partial p}{\partial \rho} \right)_T
  \]

The \( \alpha \) and \( \beta \) values for liquids are much smaller than the corresponding values for gases, so that liquids are often regarded as incompressible. Gases are susceptible to be compressed, but in a great many applications the prevailing conditions do not cause appreciable density
Perfect gas; One-Dimensional model

Changes. In those cases, the gas flow is treated as if it is incompressible. Only gas flows which experience considerable density changes will be treated in this course.

The perfect gas law will be used throughout the course to represent the equation of state for gases:

\[ p = \rho RT, \quad R = \text{specific gas constant} \]

- Basic Model: One-dimensional flow

At any cross section, the flow will be characterized by a single value of \( V \) (velocity), a single value of \( p \) (pressure), a single value of \( T \) (temperature), etc. This assumption is generally better in turbulent flow than in laminar flow, and we will be dealing almost exclusively with turbulent flows.
One-dimensional model

0 \( V \) will be associated with the mean velocity of the flow.
0 \( \dot{m} \) will denote the mass flow rate \( = \rho A V \).
0 \( \dot{M} \) will denote the rate of momentum transport \( = \dot{m} V \).

In a non-one-dimensional model:

\[
(\text{mass flow through } dA) = \rho \cdot u \cdot dA
\]
\[
\dot{m} = \int_A \rho u dA
\]

\[
(\text{momentum flow through } dA) = u \cdot \dot{m} \cdot dA
\]
\[
\dot{M} = \int_A u \cdot \dot{m} \cdot dA = \int_A \rho u^2 dA
\]

For a round pipe, we can check the validity of the 1-dimensional model by evaluating the integral definition of \( \dot{M} \) using an empirical velocity profile as input:

\( u = 1.224 V (1 - \frac{r}{R})^7 \)

The integral gives:

\[ \dot{M} = 1.02 \dot{m} V , \text{ ok to } 2\% ! \]
MASS CONSERVATION

The rate of mass flow through an area A is:

\[ \dot{m} = \rho A V \]

Along a duct whose walls are impermeable,

\[ \dot{m} = \text{constant} \]

In differential form, \( \dot{m} = \text{constant} \) is

\[ \frac{\partial \rho}{\rho} + \frac{\partial A}{A} + \frac{\partial V}{V} = 0 \]

**EXAMPLE**

Given:

- \( P_1 = 63 \text{ psia} \)
- \( T_1 = 63.55 \text{ °F} \)
- \( V_1 = 448.5 \text{ ft/sec} \)
- \( A_1 = 2.43 \text{ in}^2 \)
- \( P_2 = 55.15 \text{ psia} \)
- \( T_2 = 44.03 \text{ °F} \)
- \( A_2 = 1.816 \text{ in}^2 \)
Mass conservation

(a) Find \[ m = \frac{p_i}{RT_i} = \frac{(63)(144)}{(53.35)(63.55 + 459.7)} \]
for air \[ \frac{144}{(144 + 105) / (16.08 \times 2)} \]

\[ \rho_i = 0.3250 \text{ lbm/ft}^3 \]

\[ m = \rho_i A_i V_1 = (0.3250)(\frac{2.43}{144})(44.45) = 2.460 \text{ lbm/sec} \]

(b) Use mass conservation to find \( V_2 \)

(c) Find \( V_2 \) by Bernoulli's equation

\[ \frac{V_1^2}{2} + \frac{p_1}{\rho_i} = \frac{V_2^2}{2} + \frac{p_2}{\rho_2} \]

Tell which value (from (b) or (c)) is correct. Why?

(d) Find the error in the approximation

\[ \frac{\Delta p}{\rho} + \frac{\Delta A}{A} + \frac{\Delta V}{V} \approx 0 \]

(e) Compute \( M_1, M_2, (M_2 - M_1), \) and \( (p_i A_i - p_2 A_2) \)

(f) Compare \( (M_2 - M_1) \) and \( (p_i A_i - p_2 A_2) \). Should they be equal?
Mass conservation

**EXAMPLE**

A tank having a volume of 1 m$^3$ is filled with air whose initial pressure $P_i = 6$ atm (606.95 kPa) and initial temperature $T_i = 25^\circ C$. At $t \geq 0$, the air discharges at the rate of 0.1 m$^3$/sec. At any time during the discharge period, $p$, $\rho$, and $T$ are uniform throughout the tank. The temperature of the air is maintained constant at 25$^\circ$C. Find the pressure in the tank at $t = 5$ sec.

At any time $t$,

The mass of air in the tank = $\rho \times \text{Vol}$.

Then,

$$\frac{d(\rho \times \text{Vol})}{dt} = -\rho \times \text{Volumetric outflow}$$

$\text{Vol} = 1 \text{ m}^3$

$\text{Volumetric outflow} = 0.1 \text{ m}^3/\text{sec}$

Therefore, \[
\frac{dp}{dt} = -0.1 \rho
\]
Mass conservation

which integrates to

\[ p = p_0 e^{-0.1t} \]

But, \( p = \frac{P}{RT} \), \( p_0 = \frac{P_0}{RT} \)

and since \( T \) is a constant

\[ p = p_0 e^{-0.1t} \]

so that at \( t = 5 \) sec

\[ p = 6 e^{-0.5} = 3.64 \text{ atm} \]
ENERGY CONSERVATION

Steady state

\[(h_2 - h_1) + \frac{V_2^2 - V_1^2}{2} = Q - W\]

In the absence of shaft work

\[(h_2 - h_1) + \frac{V_2^2 - V_1^2}{2} = Q\]

When \(Q = 0\) (adiabatic)

\[h_2 + \frac{V_2^2}{2} = h_1 + \frac{V_1^2}{2} \quad \text{or} \quad h + \frac{V^2}{2} \quad \text{const.}\]

Therefore, along any flow passage where \(Q = 0\)

\[h + \frac{V^2}{2} = \text{const.}\]

At a location where \(V = 0\) (large cross sectional area),

\[h_{|V=0} = h_o = \text{const.} \quad \text{stagnation enthalpy}\]

\[\therefore \quad h + \frac{V^2}{2} = h_o\]

If we write \((h_2 - h_1) = c_p(T_2 - T_1)\)
Energy conservation

Then, \((T_2 - T_1) + \frac{V_2^2 - V_1^2}{2c_p} = \frac{Q}{c_p}\)

For adiabatic flow, \(T + \frac{V^2}{2c_p} = \text{const} = T_0\)

where \(T_0 = \text{stagnation temperature}\).

The maximum possible velocity for any adiabatic flow occurs when \(T \to 0\) (a hypothetical situation),

\[V_{\text{max}} = \sqrt{2c_p T_0}\]

EXAMPLE

\(p_1 = 39\ \text{psia},\ T_1 = 93.3\ ^\circ\text{F}\)
\(V_1 = 252.3\ \text{ft/sec}\)
\(A_1 = 1.77\ \text{in}^2\)

Rate of heat input between 1 and 2 = 8.3 \text{ Btu/sec}

Airflow: \(c_p = 0.24\ \text{Btu/lbm}^{-\circ\text{R}}\)

(a) Find \(T_0\) at state 1

\[T_{01} = T_1 + \frac{V_2^2}{2c_p} = \left(93.3 + 459.7\right) + \frac{(252.3)^2}{2(0.24)(32/17)(778)}\]

\[T_{01} = 553 + 58799\]

\[T_{01} = 558.30\ ^\circ\text{R}\]
Perfect gas: $c_p$, $c_v$, $R$, $h$, $u$

(b) What is $Q$ between 1 and 2 in units of Btu/lbm?

(c) What is the value of $T_0$ at state 2?

(d) What is the value of $T$ at state 2?

(e) Compute $V_{max}$ corresponding to $T_{01}$ and $T_{02}$

**PERFECT GAS**

\[ p = \rho RT, \quad dh = c_p dt, \quad du = c_v dt \]

\[ c_p = c_v + R \]

\[ k = \frac{c_p}{c_v} \quad (k = 1.4 \text{ for air}) \]

\[ c_p = \left( \frac{k}{k-1} \right) R \]

\[ c_v = \frac{R}{k-1} \]

When $c_p$ and $c_v$ are constant

\[ h = c_p T, \quad u = c_v T \]
Energy conservation - perfect gas

EXAMPLE

Air passing through a pipe has temperature $T$ and pressure $p$. Initially, the connection between the pipe and an adjacent tank is closed. The air in the tank is at temperature $T_1$ and pressure $P_1$. When the connection is opened, air from the pipe flows into the tank until the air pressure in the tank reaches the pipe pressure $p$. Then, an equilibrium state 2 is established in the tank, as pictured in diagram (b). The process may be assumed to be adiabatic. If the initial mass in the tank is $m_1$, what is the final mass $m_2$ in the tank?

Solution: A control volume is chosen to envelope the tank. The volume of the tank is $V_0$. The mass which enters the tank during the transient period is $(m_2 - m_1)$. 
Energy conservation - perfect gas

Let \( h \) denote the enthalpy per unit mass at the pipeline pressure \( p \) and temperature \( T \). The mass \((m_2 - m_1)\) entering the control volume carries in energy \((m_2 - m_1)h\) (this includes the transferred internal energy \((m_2 - m_1)u\) plus the flow work \((m_2 - m_1)pv\)).

The inflow raises the internal energy of the mass within the control volume. Therefore,

\[
(m_2 - m_1)h = U_{T2} - U_{T1}
\]

\[
(m_2 - m_1)h = m_2u_{T2} - m_1u_{T1}
\]

which gives

\[
m_2 = \frac{h - u_{T1}}{h - u_{T2}} m_1
\]

For a perfect gas with constant \( c_p \) and \( c_v \)

\[
m_2 = \frac{c_pT - c_vT_{T1}}{c_pT - c_vT_{T2}} m_1
\]

\( T_{T2} \) is an unknown, but it can be written as

\[
T_{T2} = \frac{P_{T2}}{\rho_{T2} R} = \frac{P}{m_2 \text{Vol} R}
\]

When this equation \( T_{T2} \) is substituted into the foregoing, the value of \( m_2 \) can be found.
Energy conservation

Differential form of the energy equation.

\[ \sum dQ \]
\[ \begin{align*}
V & \rightarrow P \\
V + dV & \rightarrow P + dp \\
A & \rightarrow A + dA
\end{align*} \]

In the steady state and in the absence of shaft work

\[ dh + d\left( \frac{V^2}{2} \right) = dQ \]

or \[ d\left( h + \frac{V^2}{2} \right) = dQ \] or \[ dh_o = dQ \]

Substitute \( dh = c_p dT \)

\[ c_p dT + d\left( \frac{V^2}{2} \right) = dQ \]

or \[ d\left( T + \frac{V^2}{2c_p} \right) = \frac{dQ}{c_p} \] or \[ dT_o = \frac{dQ}{c_p} \]

If \( dQ = 0 \), then \( dh_o = 0 \) \( dT_o = 0 \)
Every process involving fluid flow and heat transfer is, to some degree, irreversible. Therefore, all real processes are marked by a change of entropy. However, there are processes where the entropy changes are so small that they can be neglected. Such processes are called isentropic.

Suppose, for example, that a gas experiences a very rapid reduction in flow cross section, as pictured in the diagram. The pressure drop between stations 1 and 2 is the sum of the irreversible friction-related losses at the duct wall and the reversible pressure decrease which occurs to compensate the increase of momentum. If the latter is much larger than the former, than the friction-related part can be neglected. Also, the heat transfer can be neglected if \( Q < \frac{1}{2} (V_2^2 - V_1^2) \) (see page 10). Therefore, such flows can be regarded as isentropic.
EQUATIONS OF ENTROPY CHANGE

1st Tds equation:
\[ Tds = du + pdv \]
\[ du = cvdT \quad p/T = \rho R = R/\nu \]
\[ ds = cvdT + R d\nu/k \]
\[ S_2 - S_1 = cv \ln \left( \frac{T_2}{T_1} \right) + R \ln \left( \frac{\nu_2}{\nu_1} \right) = R \left[ \frac{c_v}{R} \ln \left( \frac{T_2}{T_1} \right) + \ln \left( \frac{\nu_2}{\nu_1} \right) \right] \]
\[ S_2 - S_1 = R \ln \left[ \left( \frac{T_2}{T_1} \right)^{k-1} \left( \frac{\nu_2}{\nu_1} \right) \right] \]

Isentropic:
\[ S_2 - S_1 = 0 \quad \Rightarrow \quad T^{k-1} \nu = \text{constant} \]
\[ \text{or} \quad T/\nu^{k-1} = \text{constant} \]

2nd Tds equation:
\[ Tds = dh - \nu dp \]
\[ S_2 - S_1 = R \ln \left[ \left( \frac{T_2}{T_1} \right)^{k-1} \left( \frac{\nu_2}{\nu_1} \right) \right] \]

Isentropic:
\[ T^{k/(k-1)}/\nu = \text{constant} \]

Another isentropic relation follows from the foregoing by using the perfect gas law:
\[ p/\rho^k = \text{constant} \]
MOMENTUM CONSERVATION

Newton's 2nd Law for a discrete particle of mass is \( F = ma = m(dV/dt) = d(mV)/dt = d(\text{momentum})/dt \). In other words, the net force \( F \) is equal to the rate of change of momentum. This law applies separately in each coordinate direction.

For a fluid, a control volume approach is used, and Newton's Law (called the conservation of momentum principle) takes the form:

\[
\text{Net rate of outflow of momentum from CV} = \text{Net force on surface of CV}
\]

This statement applies in each coordinate direction and in the steady state.

Consider first a duct of constant cross section.

\[\begin{align*}
\text{Control volume (CV)} & \quad \text{dA}_w \\
A & \quad A + dp \\
V & \quad V + dV \\
p & \quad p + dp
\end{align*}\]
Momentum conservation

rate of momentum inflow = \( mV \)
rate of momentum outflow = \( m(V + dV) \)

Net rate of momentum outflow = \( m dV \)
But \( m = \rho AV \), so that
Net rate of momentum outflow = \( \rho AV dV \)

Forces: \( P \)
\( p \)
\( p + dp \)

Shear \( \tau \)
\( \tau \) is the retarding shear stress exerted by the wall on the fluid

Net force in \( +x \) direction
\[
 pA = (p + dp)A - \tau dA_w \\
= -Adp - \tau dA_w
\]

Conservation of momentum:
\[
\rho AV dV = -Adp - \tau dA_w
\]
\[
dA_w = Cdx \quad (C = \text{circumference})
\]
\[
D_h = \text{hydraulic diameter} = \frac{4A}{C}
\]
Momentum conservation

Friction factor \( f = \frac{1}{2} \rho v^2 \)

Substitution of \( dA_w \), \( D_m \), and \( f \) yields

\[ \rho V dV = -dP - 4f \left( \frac{1}{2} \rho v^2 \right) \frac{dx}{D_m} \]

- Next, consider a duct whose cross sectional area \( A \) varies with position along the duct.

Net outflow of momentum

\[ = m(V + dV) - mV = m dv = \rho AV dV \]

Net force in the \( +x \) direction

\[ = pA - (p + dp)(A + dA) + (p + \frac{1}{2} dp)dA_w \sin \alpha \]

To project in \( x \) direction

\[ \text{Force} = T dA_w \cos \alpha \]

Projected area
Momentum conservation

Note that \( dA_w \sin \alpha = dA \)
\[
dA_w \cos \alpha = C \, dx
\]
\[C = \text{circumference}\]

Momentum conservation:

\[
\rho AV \, dV = \rho \frac{A}{A} - \rho \frac{A}{A} - \rho dA - Adp - dpdA + \rho \frac{A}{A} + \frac{1}{2} dpdA - \tau C \, dx
\]

\[
\rho AV \, dV = - Adp \left( 1 + \frac{1}{2} \frac{dA}{A} \right) - \tau C \, dx
\]

Substitute \( f = \frac{\tau}{\frac{1}{2} \rho v^2} \), \( D_H = \frac{4A}{C} \)

\[
\rho v \, dV = - dp - 4f \left( \frac{1}{2} \rho v^2 \right) \frac{dx}{D_H}
\]

This equation is identical to the one at the top of page 18 for the duct of constant cross section.
Momentum conservation

**EXAMPLE**

An air jet discharges from an isentropic converging nozzle and impinges on a stationary blade which turns the flow as shown in the figure. The jet may be assumed not to spread. The quantities $F_x$ and $F_y$ are the forces, whose magnitudes have to be found, which hold the blade in place.

**Given data:**
- $P_1 = 1.5 \text{ atm}$, $T_1 = 308 \text{ K}$, $V_1 = 60 \text{ m/sec}$
- $A_1 = 25 \times 10^{-4} \text{ m}^2$, $P_2 = 1 \text{ atm} = 1.01325 \times 10^5 \frac{\text{N}}{\text{m}^2}$

**Solution:** Since the nozzle is isentropic

$$T_2^{\frac{K-1}{K}} / P = \text{constant} \quad \text{(page 16)}$$

or

$$T / P^{\frac{K-1}{K}} = \text{constant}$$

so that

$$T_2 = T_1 \left( \frac{P_2}{P_1} \right)^{\frac{K-1}{K}} = 308 \left( \frac{1}{1.5} \right)^{\frac{1.4-1.0}{1.4}} = 274.3 \text{ K}$$
Momentum conservation

The mass flow \( m \) is constant if the entrainment of ambient air into the jet is neglected (equivalent to no spreading of the jet). The most convenient location to compute \( m \) is at station 1.

\[
m = \rho A_1 V_1 = \frac{P_1}{RT_1} A_1 V_1 = \left( \frac{1.5}{287.1} \frac{1.013 \times 10^5}{308} \right) \text{ m} \cdot \text{kg/m} \cdot \text{s} \cdot \text{K} \cdot \text{m} \cdot \text{s}^{-1} = (25 \times 10^{-4}) \text{ (60)}
\]

\[
m = 0.258 \text{ kg/sec} \quad R: \left[ \frac{J}{\text{kg} \cdot \text{K}} \right], \quad (1 \text{ J} = 1 \text{ Nm})
\]

Since the nozzle flow is also adiabatic

\[
T + \frac{V^2}{2c_p} = \text{constant} \quad \text{(page 10)}
\]

or

\[
V_2 = \sqrt{2c_p (T_1 - T_2) + V_1^2}
\]

For air: \( c_p = 1.0035 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} = 1003.5 \frac{\text{J}}{\text{kg} \cdot \text{K}} \)

so that

\[
V_2 = \sqrt{(2)(1003.5)(308 - 274.3) + (60)^2}
\]

\[
V_2 = 266.5 \text{ m/s} \quad (1 \text{ J} = 1 \frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2})
\]

The assumption of non-spreading of the jet leads to

\[
V_3 = 266.5 \text{ m/sec}
\]
Momentum conservation

Next, the momentum conservation principle (page 17) is applied separately in the x- and y-directions. For the x-direction:

\[ x \text{-direction momentum outflow rate} \]
\[ - x \text{-direction momentum inflow rate} \]
\[ = \text{Net applied force in the x-direction} \]

The outflow term is \( \dot{m}V_{3x} \), and

\[ V_{3x} = - V_3 \cos 30^\circ = -(266.5)(0.866) \]
\[ V_{3x} = -230.8 \]

The minus sign signifies that \( V_{3x} \) is in the minus x-direction. The inflow term is \( \dot{m}V_{2x} = \dot{m}V_2 \). Therefore,

\[ \dot{m}(-230.8 - 266.5) = F_x \]
\[ F_x = -128.3 \text{N} \] (minus means minus x-direction)

Similarly,

\[ F_y = \dot{m}(V_{3y} - V_{2y}) = (0.258)(266.5)\sin 30^\circ \]
\[ F_y = 34.4 \text{ N} \]
VELOCITY OF SOUND

Finite pressure disturbance moving into a still fluid:

Still Air

\[ \begin{align*}
\text{p} + \Delta \text{p} & \quad \text{p} \\
\text{p} + \Delta \text{p} & \quad \text{p} \\
\text{T} + \Delta \text{T} & \quad \text{T} \\
V = \Delta V & \quad V = 0
\end{align*} \]

Moving front

For an observer riding on the moving front:

Stationary front

\[ \begin{align*}
\text{p} + \Delta \text{p} & \quad \text{p} \\
\text{p} + \Delta \text{p} & \quad \text{p} \\
\text{T} + \Delta \text{T} & \quad \text{T} \\
V = c - \Delta V & \quad V = c
\end{align*} \]

Mass conservation:

\[ \rho Ac = (\rho + \Delta \rho)(A)(c - \Delta V) \]

\[ \Delta V = c \left( \frac{\Delta \rho}{\rho + \Delta \rho} \right) \rightarrow \Delta V < c \]

If \( \Delta \rho \ll \rho \rightarrow \Delta V \ll c \), the induced fluid velocity is much smaller than the pressure wave velocity.
Sound velocity

**Momentum conservation:**
\[ m(c - \Delta v) - mc = pA - (p + \Delta p)A \]
\[ \Delta p = \rho c \Delta v \]

Eliminate \( \Delta V \) and get:
\[ c^2 = \frac{\Delta p}{\Delta \rho} \left( \frac{p + \Delta p}{\rho} \right) \]

**Sound wave:** \( \Delta p/\rho \ll 1 \), \( \Delta p/\Delta \rho \to dp/d\rho \)

\[ \rightarrow \quad c = \sqrt{dp/d\rho} \]

Should \( dp/d\rho \) be evaluated at constant \( T \) or constant \( s \)?

This is correct.

\[ (dp/d\rho)_T = RT, \quad (dp/d\rho)_s = kRT \]

\[ \rho = \text{const} + \rho k \]

\[ c = \sqrt{RT} \]

**For air,** \( k = 1.4 \), \( R = 53.35 \) \( \frac{ft \cdot lb_f}{lbm \cdot \circ R} = 287.1 \) \( \frac{J}{kg \cdot \circ K} \)

\[ c = \sqrt{(1.4)(53.35)(32.17)} \frac{T}{T} = 49.02 \sqrt{\frac{T}{\circ R}} \]

\[ \text{Foot/sec} \]

\[ c = \sqrt{(1.4)(287.1)T} = 20.05 \sqrt{\frac{T}{\circ K}} \]

\[ \text{m/s} \]
Sound velocity

The foregoing was focused on gases.

For liquids \( \frac{dp}{d\rho} = \frac{K}{\rho} \), \( K = \text{bulk modulus} \)

so that \( c = \sqrt{\frac{K}{\rho}} \)

**Homework problem**

Calculate \( c \) in m/sec and ft/sec at 300°K for the following

(a) air

(b) helium

(c) hydrogen

(d) carbon dioxide

(e) carbon monoxide

(f) liquid water \( (K = 3.20 \times 10^5 \text{ lb}_f/\text{in}^2) \)

(g) steam
**MACH NUMBER**

\[ M = \frac{V}{C} \quad \text{V and C correspond to the same location} \]

- \( M < 1 \) "incompressible" flow
- \( M < 1 \) subsonic flow
- \( M \approx 1 \) transonic flow
- \( M > 1 \) supersonic flow
- \( M \gg 1 \) hypersonic flow

For a gas: \( M = \frac{V}{\sqrt{kRT}} \) or \( V = M \sqrt{kRT} \)

Introduce \( M \) into basic equations:

\[ T_0 = T + \frac{V^2}{2c_p} = T \left( 1 + \frac{V^2}{2c_p T} \right) = T \left( 1 + \frac{V^2}{kRT \frac{kR}{R}} \right) \]

\[ T_0 = T \left( 1 + \frac{k-1}{2} M^2 \right) \]

\[ \dot{m} = \rho A V = \left( \frac{p}{RT} \right) A V = \rho A \frac{V}{\sqrt{kRT}} \sqrt{\frac{k}{RT}} \]

\[ \dot{m} = pA M \sqrt{\frac{k}{RT_0}} \sqrt{\frac{T_0}{T}} \]

\[ \dot{m} = pA M \sqrt{\frac{k}{RT_0}} \sqrt{1 + \frac{k-1}{2} M^2} \]

"pam" equation
PRESSURE WAVE PATTERNS

The discussion which follows will demonstrate the different sound wave patterns which occur when the speed $V$ of a moving source of sound is either $= 0$, $< c$, $= c$, or $> c$, where $c$ is the velocity of sound. Suppose, for concreteness, the source "emits" a spherical sound wave at 1-second intervals.

(a) Source velocity $V = 0$

The diagram shows the wave pattern at $t = 3$ sec. The wave emitted at $t = 0$ is a
Pressure wave patterns

sphere of radius $3c$; the wave emitted at $t=1$ sec is a sphere of radius $2c$; etc. The sound field is symmetric.

(b) $V < c$

The diagram shows that the sound field outruns the moving source. This means that sound source is always immersed in its own sound. Thus, for example, a passenger in a subsonic airplane will hear the sound of the motors no matter where in the plane he/she is seated. [Advertisements for the Caravelle, the first jet airliner with rear-mounted engines, claimed that passengers seated forward of the engines would get a quiet ride.]
When \( V = c \), pressure waves are not present ahead of the moving source. Note that all emitted pressure waves are tangent to a plane which is perpendicular to the direction of motion and passes through the source. The region upstream of this plane is called the zone of silence, while the region downstream of the plane is called the zone of action. Note that the communication of pressure information is confined to the zone of action.

(d) \( V > c \)
In this case, the source outruns the sound field. Therefore, a passenger on a supersonic airliner who is seated ahead of the engines will enjoy a quiet ride. A cone - the Mach cone - can be constructed which separates the zone of action from the zone of silence. The Mach cone \( \alpha \) is shown in the diagram. It is given by

\[
\sin \alpha = \frac{c}{V} = \frac{1}{M}
\]

There is a significant change of pressure across the surface of the Mach cone, which
Pressure wave patterns gives rise to a significant change of density. An abrupt change of density in a flowing fluid can be visualized by well-established optical methods. This enables the Mach cone angle $\alpha$ to be measured, and from that measurement

$$M = \frac{1}{\sin \alpha}$$

This provides a method for determining the Mach number.

**EXAMPLE:**

![Diagram of Mach cone and aircraft](image.png)
Pressure wave patterns

An airplane travelling at \( M = 1.5 \) passes over an observer situated on the ground. The plane is 1000 m above the ground, and the ambient temperature is 20°C. In how many seconds will the observer hear the aircraft?

Solution: The sound velocity \( C \) corresponding to \( T = 20 + 273.15K \) is found by using the formula at the bottom of page 25:

\[
C = 20.05\sqrt{T} = 20.05\sqrt{293.15}
\]
\[
C = 343.3 \text{ m/sec}
\]

From page 32,
\[
\alpha = \sin^{-1} \left( \frac{1}{M} \right) = \sin^{-1} \left( \frac{1}{1.5} \right)
\]
\[
\alpha = 0.7297 \text{ radians} = 41.8^\circ
\]

Next, from the diagram on page 32,
\[
\tan \alpha = \frac{Z}{Vt}
\]
\[
t = \frac{Z}{V \tan \alpha} = \frac{1000}{(1.5)(343.3) \tan 41.8^\circ} = 2.17 \text{ sec}
\]
ISENTROPIC FLOW CONCEPTS

Large and rapid area change causes changes in pressure which far exceed those caused by friction. Kinetic energy changes far exceed the external heat transfer Q. Therefore, although the flow is not strictly reversible and adiabatic, the entropy change can be neglected. So, the flow will be treated as isentropic.

Qualitative features:

Mass conservation
\[ \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \]  
(page 5)

Momentum conservation (no-friction)
\[ \rho V dV = -dp \]  
(page 20)

Eliminate \( dV \)
\[ \frac{d\rho}{\rho} + \frac{dA}{A} - \frac{dp}{\rho V^2} = 0 \]  
(page 23)

For an isentropic flow:
\[ c^2 = \frac{dp}{d\rho} \text{ or } dp = \frac{d\rho}{c^2} \]

Also, \( M = \frac{V}{c} \)

So that
Isentropic Flow

\[ \frac{dp}{dA} = \rho V^2 \frac{1}{A (1 - M^2)} \]

and \( dV = - \frac{dp}{\rho V} \), and so

\[ \frac{dV}{dA} = - \frac{V}{A} \frac{1}{(1 - M^2)} \]

- Response of \( p \) and \( V \) to area change:

(a) \( M < 1 \) \( \rightarrow (1 - M^2) > 0 \)

\[ \frac{dp}{dA} > 0, \quad \frac{dV}{dA} < 0 \]

\( dA > 0 \) \( \rightarrow \), \( dA < 0 \) \( \rightarrow \)

\( p \) increases \( \rightarrow \), \( V \) decreases \( \rightarrow \)

(b) \( M > 1 \) \( \rightarrow (1 - M^2) < 0 \)

\[ \frac{dp}{dA} < 0, \quad \frac{dV}{dA} > 0 \]

\( dA > 0 \) \( \rightarrow \), \( dA < 0 \) \( \rightarrow \)

\{ \( p \) decreases \}\( \rightarrow \) \{ \( p \) increases \}\( \rightarrow \)

\{ \( V \) increases \}\( \rightarrow \) \{ \( V \) decreases \}\( \rightarrow \)
Isentropic flow

**Concept of minimum area**

Consider V vs A diagram for the converging nozzle. Start at the stagnation state. As A decreases in the flow direction, V increases (i.e., the flow accelerates). The acceleration continues until \( M = 1 \), where \( \frac{dV}{dA} = \infty \) (see graph). Then, it remains to consider what happens after that.

There are four possible scenarios, given schematically by the four paths \( \text{I, II, III, and IV} \) in the figure. From the figure,

<table>
<thead>
<tr>
<th>Path</th>
<th>( \frac{dV}{dA} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>II</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>III</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>IV</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>
Isentropic flow

On the other hand, from the physics of the problem as represented by

\[
\frac{dV}{dA} = -\frac{V}{A} \frac{1}{1-M^2}
\]

<table>
<thead>
<tr>
<th>Path</th>
<th>(\frac{dV}{dA})</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(&gt;0)</td>
</tr>
<tr>
<td>II</td>
<td>(&lt;0)</td>
</tr>
<tr>
<td>III</td>
<td>(&lt;0)</td>
</tr>
<tr>
<td>IV</td>
<td>(&gt;0)</td>
</tr>
</tbody>
</table>

By comparing the tables, it is seen that only paths I and II are possible because they obey the physical laws. Note that the cross-sectional area increases along both paths I and II.

Therefore, if a flow is accelerated to \(M=1\) through a converging nozzle, then downstream of the section where \(M=1\), the cross section must enlarge if the flow is steady and isentropic. Thus, if \(M=1\) is attained, the cross-sectional area at that location is a minimum. Note, however, that the smallest cross-sectional area is not necessarily a location where \(M=1\) (to be elaborated shortly).
Isentropic flow

**ISENTROPIC FLOW EQUATIONS**

(From page 27)

\[
\frac{T}{T_0} = \frac{k}{k-1} \left( 1 + \frac{k-1}{2} M^2 \right) \quad \text{(Isentropic is adiabatic)}
\]

\[
p = \text{const} \cdot T^{k/(k-1)} \quad \text{(page 16)}
\]

\[
p = p_0 \left( 1 + \frac{k-1}{2} M^2 \right)^{k/(k-1)}
\]

\[
\rho = \text{const} \cdot p^{\frac{k}{k-1}} \quad \text{(page 16)}
\]

\[
\rho = \rho_0 \left( 1 + \frac{k-1}{2} M^2 \right)^{1/(k-1)}
\]

\[
k = 1.4 \quad \frac{T}{T_0} = \frac{1}{1 + 0.2 M^2}, \quad \frac{p}{p_0} = \left( 1 + 0.2 M^2 \right)^{3.5}
\]

Eliminate \( p \) from \( pAM \) equation:

(From page 27 and \( p/p_0 \) above)

\[
m = \rho_0 AM \sqrt{\frac{k}{RT_0}} \left[ 1 + \frac{k-1}{2} M^2 \right]^{(k+1)/(2(k-1))}
\]

or

\[
\frac{A}{\sqrt{\rho_0 RT_0}} \left[ \frac{m}{\rho_0} \right]^{\frac{1}{k}} = \left[ 1 + \frac{k-1}{2} M^2 \right]^{\frac{(k+1)}{2(k-1)} M}
\]

Constant throughout flow passage

Therefore, the foregoing equation relates the variation of \( M \) to the variation of \( A \).

For small \( M \), \( \text{RHS} \approx \frac{1}{M} \)
Isentropic flow

For large $M$, \( \text{RHS} \approx M^{2(k-1)} \)

\[
\frac{A}{\frac{m}{P_0} \sqrt{\frac{RT_0}{k}}} = \text{constant}
\]

If $M = 1$ is attained in a flow passage of variable cross sectional area, then it is attained at the minimum area.

Therefore:
- $M = 1 \rightarrow$ minimum area
- Minimum area is not necessarily an indication of $M = 1$

Let $A^*$ denote the area $A$ when $M = 1$

\[
A^* = \frac{m}{P_0} \sqrt{\frac{RT_0}{k}} \left( \frac{k+1}{2} \right)^{(k+1)/(2(k-1))}
\]

\[
\frac{A}{A^*} = \frac{1}{M} \left[ \frac{1 + \frac{k-1}{2}M^2}{(k+1/2)} \right]^{k+1/(2(k-1))}
\]

For $k = 1.4$:

\[
A^* = 1.4604 \frac{m}{P_0} \sqrt{RT_0} = \text{any units}
\]

\[
\text{Ibm/Sec} = \sqrt{\frac{1}{T_0}} \cdot \frac{1}{0.5317 \text{ Ps}} \cdot (\text{English units} \cdot A^* \text{ in} \text{ in}^2)
\]
Isentropic flow

The $A/A^*$ equation can be plotted as

\[
\frac{A}{A^*}
\]

For known values of $m$, $T_0$, and $P_0$, the value of $A^*$ can be calculated from page 39. If the actual minimum area $A_{\text{min}}$ of a nozzle is greater than $A^*$, then $M \neq 1$ at that minimum area. Here are two examples:

1.  
   \[
   \begin{array}{c}
   \text{M} \\
   \text{1.0} \\
   \text{O} \ 	ext{X}
   \end{array}
   \]

2.  
   \[
   \begin{array}{c}
   \text{M} \\
   \text{1.0} \\
   \text{X}
   \end{array}
   \]
Isentropic flow

The symbol * is used to denote the $M = 1$ state for isentropic flow in a passage of varying cross sectional area. From page 39

$$A^* = 1.4604 \frac{m}{p_0} \sqrt{RT_0}$$

and from page 38 in which $M$ is set equal to 1.0

$$\frac{T^*}{T_0} = 0.8333$$
$$\frac{p^*}{p_0} = 0.5283$$
$$\frac{\rho^*}{\rho_0} = 0.6339$$

Special note should be taken of the ratio $\frac{p^*}{p_0} = 0.5283$ as an identifier of the attainment of the sonic state.

Although the equations connecting $T/T_0$, $p/p_0$, $\rho/\rho_0$ (page 38), and $A/A^*$ (page 39) with $M$ are rather simply evaluated with the aid of a pocket calculator, it is convenient to list these (and other ratios) in a table to facilitate computation. That table is called Table 132 and is included at the end of
Isentropic flow

This section of the notes, i.e., page 64. The tabulated numbers apply specifically for gases with \( k = 1.4 \).

The table heading is

\[
\begin{array}{ccccccc}
M & M^* & \frac{A}{A^*} & \frac{P}{P_0} & \frac{\rho}{\rho_0} & \frac{T}{T_0} & \frac{F}{F^*} & \frac{A_p}{A^*_p} \\
0 & - & - & - & - & - & - & -
\end{array}
\]

0.34 - 1.8229 0.92312 - - -

The table is based on the Mach number \( M \) playing the role of the independent variable, and \( M \) is assigned values of 0, 0.01, 0.02, 0.03 ... A line of the table is displayed in order to explain how the table is used. Suppose that during a calculation, the \( P/P_0 \) ratio of 0.9231 is encountered, and it is
Isentropic flow

desired to find the $A/A^*$. Clearly, $A/A^*$ can be read directly from the table (page 42) as 1.8229. Without the table, a two-step calculation process would be necessary.

Step 1: Compute $M$ from

$$0.92312 = \frac{1}{(1 + \frac{k-1}{2} M^2)^{\frac{k}{k-1}}} \quad (k = 1.4)$$

Step 2: With the $M$ value just found, compute $A/A^*$ from

$$\frac{A}{A^*} = \frac{1}{M} \left[ 1 + \frac{k-1}{k+1} \frac{M^2}{2} \right]^{\frac{k+1}{2(k-1)}} \quad (k = 1.4)$$

It is clear that the table offers advantages for certain types of calculations. The table entries are sufficiently dense that linear interpolation is acceptable.

A graphical presentation of several of the tabulated quantities is made on page 44 as a function of the Mach number. As seen there, $p/p_o$, $p/p_o$, and $T/T_o$ decrease monotonically with $M$. $A/A^*$ is,
Isentropic flow

on the other hand, double-valued; that is,
Isentropic flow

for a given value of $A/A^*$, there are two possible values of $M$ — one subsonic and the other supersonic.

The table lists quantities such as $M^*$, $F/\dot{F}^*$, and $Ap/A^*p_0$ which have not yet been discussed. The quantity $M^*$ is used as a possible replacement for $M$. Note that $M$, which is defined as

$$M = \frac{V}{\sqrt{KRT}}$$

for a perfect gas, is not a direct reflection of the velocity $V$; that is, variations of $M$ are not caused solely by variations of $V$. Rather, $M$ varies in response to both $V$ and $T$. On the other hand, $M^*$ is purely a reflection of $V$, according to its definition

$$M^* = \frac{V}{\sqrt{KRT^*}}$$

where $T^*$ is the value of $T$ at the $M=1$ state. For air, in English units,
Isentropic flow

units

\[ M^* = \frac{V}{49.02/ T^*} \]

or, since \( T^* = 0.8333 T_0 \) for \( k = 1.4 \), then

\[ M^* = \frac{V}{44.75/ T_0} \]

The relationship between \( M \) and \( M^* \) for \( k = 1.4 \) is shown in the graph.

Note that \( M^* \) remains finite as \( M \to 0 \). This occurs because \( V \) stays finite (finite \( M^* \)), but \( T \to 0 \) (infinite \( M \) ).
Isentropic flow

The last column of Table B2, \( \frac{A_p}{A^*p_0} \), is actually a merging of the \( \frac{A}{A^*} \) and \( \frac{p}{p_0} \) columns. It is, however, a very useful column of numbers in problem solving. The key to its importance comes from the equation (bottom of page 39), i.e.,

\[
\frac{m\sqrt{T_0}}{0.5317} = A^*p_0
\]

This equation indicates that when \( m \) and \( T_0 \) are constant, the product \( A^*p_0 \) is also a constant. The implementation of this finding will be described later.

The remaining column of Table B2 that needs elucidation is the \( \frac{F}{F^*} \) column. \( F \) is called the impulse function (sometimes called \( I \)). To motivate the definition of \( F \), consider a momentum balance on a flow passage of arbitrary shape.

[Diagram of flow passage with vectors indicating flow direction and sections labeled 1 and 2.]
Isentropic flow

For the x-direction,

\[ \dot{m} (V_2 - V_1) = p_1 A_1 - p_2 A_2 + \tau \]

where \( \tau \), the thrust, is the net force exerted in the +x direction by the walls and obstructions on the fluid; in general, \( \tau \) is an unknown. With \( \dot{m} = p_1 A_1 V_1 = p_2 A_2 V_2 \),

\[ (p_2 A_2 V_2) V_2 - (p_1 A_1 V_1) V_1 = p_1 A_1 - p_2 A_2 + \tau \]

The first two terms involve the product \( \rho V^2 \). For a perfect gas,

\[ \rho V^2 = \frac{PV}{RT} \]

so that

\[ \frac{PV}{RT} = k \rho \frac{V^2}{RT} = k \rho M^2 \]

so that

\[ k p_2 A_2 M_2^2 - k p_1 A_1 M_1^2 = p_1 A_1 - p_2 A_2 + \tau \]

or

\[ p_2 A_2 (1 + k M_2^2) - p_1 A_1 (1 + k M_1^2) = \tau \]

If \( F \) is defined as

\[ F = A (\rho + \rho V^2) = pA (1 + k M^2) \]
Isentropic flow

\[ \phi_T = F_2 - F_1 \]

Therefore, the thrust \( \phi_T \) can be determined as the difference in \( F \) values. The \( F/F^* \) ratio in Table B2 was tabulated from

\[ \frac{F}{F^*} = \frac{pA(1+kM^2)}{p^*A^*(1+k)} = \frac{Ap}{A^*p_0} \frac{1}{p^*/p_0} \frac{1+kM^2}{\text{fct of } M} \]

\( \frac{\text{fct of } M}{\text{const}} \)

**EXAMPLE**

\( A_1 = 2.25 \text{ in}^2 \)
\( A_2 = 1.80 \text{ in}^2 \)
\( p_1 = ? \)
\( p_2 = 94.8 \text{ psia} \)

From page 47, \( A^*_p = \frac{m\sqrt{T_0}}{0.5317} = \frac{4.1\sqrt{540}}{0.5317} = 179.2 \text{ lb}_f \)

\[ \frac{A_2p_2}{A^*_p} = \frac{(1.80)(94.8)}{179.2} = 0.9522 \]

\( A_2/A^* \quad B_2 \quad P_2/P_0 \quad M_2 \quad \)

\( A_2 = 1.80 \text{ in}^2 \)
\( A^*_p = \frac{m\sqrt{T_0}}{0.5317} = \frac{4.1\sqrt{540}}{0.5317} = 179.2 \text{ lb}_f \)
\( P_2 = 94.8 \text{ psia} \)
Example:

Two operating modes: mode I and mode II. $p_0$ and $T_0$ are the same for modes I and II. Specific data:

$m_1 = 5.11 \text{ lb/sec}$ \hspace{1cm} $m_2 = 3.16 \text{ lb/sec}$

$(p_1)_I = 82.12 \text{ psia}$ \hspace{1cm} $(p_2)_I = 101.5 \text{ psia}$

$T_0 = 500^\circ F$

Find $(M_1)_I$ and $(M_2)_II$

Note that $p_0 A^* = \frac{m}{\sqrt{T_0}} / 0.532$, so that

$$\frac{(p_1 A_1)}{(p_0 A^*)_I} = \frac{82.12 A_1}{5.11 / 500 / 0.532} = 0.3823 A_1$$

$$\frac{(p_1 A_1)}{(p_0 A^*)_II} = \frac{101.5 A_1}{3.16 / 500 / 0.532} = 0.7642 A_1$$

Guess $p_0$ and ratio to find $(\frac{p_1}{p_0})_I$

If this value of $p_0$ does not agree with the guessed value, repeat the process.