Determining Path Trajectories:

Interpolation and Functions for Describing Motion between Consecutive Specified Positions


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Typical form of equation for determining robot manipulator’s mounting flange position in H.T.

\[ T_6 = (\text{coord}) (\text{pos}) (\text{tool})^{-1} \]

Where

\textbf{coord}: Transformation for working coordinate frame

\textbf{pos}: Transform for desired position of tool tip or object

\textbf{tool}: Transform for describing the tool tip or object to be moved

The above (typically used for calculating needed manipulator joint “angles”) may also be re-written in alternate form:

\[ T_6(\text{tool}) = (\text{coord}) (\text{pos}) \]

By specifying \textbf{tool} and \textbf{coord}, all motion instructions can be written in terms of the desired position, \textbf{pos}
The Manipulator Path:

Motion between positions

Consider series of positions:

Posn. 1: \((T_6)_1 \times \text{Tool}_1 = \text{Coord}_1 \times \text{Pos}_1\)

Posn. 2: \((T_6)_2 \times \text{Tool}_2 = \text{Coord}_2 \times \text{Pos}_2\)

Posn. 3: \((T_6)_3 \times \text{Tool}_3 = \text{Coord}_3 \times \text{Pos}_3\)

Two approaches: A. Joint Coordinates

B. World or Cartesian coordinates

A. Joint Coordinates:
   Solve above equations for \(T_6\)
   Calc. \(\Delta\) (Joint coordinates)
   Control joint trajectories

B. Cartesian Coordinates:
   Calculate distance and direction between consecutive positions
   Calculate orientation change between two positions
   Control trajectory in world coordinates
Consider previous example, need to get a continuous motion/trajectory between PD and PHA across different working coord systems

Task Program:

1. Tool = E PG\(^{-1}\) → Attach tool
2. Coord = Z\(^{-1}\)P → Set up coord w.r.t pin
3. Move PA → \(T_6 = \text{coord} \times \text{PA} \times (\text{tool})^{-1}\)
4. Move Origin → \(T_6 = \text{coord} \times (\text{tool})^{-1}\)
5. Grasp
6. Move PD → \(T_6 = \text{coord} \times \text{PD} \times (\text{tool})^{-1}\)
7. Coord = Z\(^{-1}\)BH\(_i\) → Set up coord w.r.t hole
8. Move PHA → \(T_6 = \text{coord} \times \text{PHA} \times (\text{tool})^{-1}\)
9. Move PIN → \(T_6 = \text{coord} \times \text{PIN} \times (\text{tool})^{-1}\)
10. Release
11. Move PHA → Move straight out w.r.t hole axis
    → \(T_6 = \text{coord} \times \text{PHA} \times (\text{tool})^{-1}\)
12. Go to 1 → Back to instruction #1
Common to both approaches, use procedure

Given: Desired velocity

Can obtain: Time period $T_2$ to move from position (i) to position (i + 1)

Then define: Time $t$ to run from 0 to $T_i$ between each pair of positions

- Must redefine present position in terms of subsequent coordinate systems and tool definitions.
- Can also capture motion in continuous variable. For example, consider a moving conveyor belt

 Present position...

At posn 1: $^1T_6 \times Tool_1 = Coord_1(t=0) \ 1^1Pos_1$

Evaluated at $t=0$

(Beginning of motion from position 1 to position 2)

- For each position, “Look ahead.” Define a second transformation in terms of the next coordinate system
We rewrite the expression for position 1 in terms of the coordinate and tool transformations of position 2 -- $^2\text{Pos}_1$

$$^1T_6 \times \text{TOOL}_2 = \text{COORD}_2(t=0) \times ^2\text{Pos}_1$$

Note: $^1T_6$ is the same for both equations above. We can now solve for $^2\text{Pos}_1$

$$^2\text{Pos}_1 = \text{COORD}_2^{-1}(t=0) \times ^1T_6 \times \text{TOOL}_2$$

$$= \text{COORD}_2^{-1}(t=0) \times \text{COORD}_1(t=0) \times ^1\text{POS}_1 \times \text{TOOL}_1^{-1} \times \text{TOOL}_2$$

Thus the motion between any two positions $i$ and $(i + 1)$ can be expressed as a motion from

$$^6T_6 = \text{COORD}_2(s) \times ^2\text{POS}_1 \times \text{TOOL}_2^{-1}$$

TO

$$^6T_6 = \text{COORD}_2(s) \times ^2\text{POS}_2 \times \text{TOOL}_2^{-1}$$

Note: Coord is a function of $s$, a moving coordinate system variable
For path determination, we changed $^2\text{POS}_1$ to $^2\text{POS}_2$

in a controlled manner.

**Result:**

Motion of gripper now accounts for change in working **coord** system **AND** is independent of $s$ (which can also be used for example to handle conveyor belt motion).

We wish to calculate how this change should take place?

**Need:** Continuity of position

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There are many possible path trajectories:

Consider B as via point (used as guide):
Given: Initial and final position, constant velocities at beginning and end of motion, and constant acceleration for finite time.

Can solve using:

\[ q = a_2 t^2 + a_1 t + a_0. \]

Result:

\[ q = \left[ (\Delta C \frac{t_a}{T} + \Delta B) h - 2\Delta B \right] h + \Delta B \]
\[ \dot{q} = \frac{1}{T_a} \left( \Delta C \frac{t_a}{T} + \Delta B \right) h - \frac{\Delta B}{T_a} \]
\[ \ddot{q} = \frac{1}{2T_a} \left( \Delta C \frac{t_a}{T} + \Delta B \right) \]

\text{WHERE} \quad h = \frac{T + t_a}{2T_a}

Relatively straightforward equations

However,…..

Note the discontinuity in acceleration
An alternate path with different constraints:

Displacement

Velocity

Acceleration
Given: Position, velocity and acceleration boundary conditions for multiple points in trajectory. Due to symmetry...

\[
q = a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0
\]

We get

\[
\begin{cases}
q = \left[ (\Delta C \frac{T_{acc}}{T_1} + \Delta B) (2-h)h^2 - 2\Delta B \right] h + B + \Delta B \\
\dot{q} = \left[ (\Delta C \frac{T_{acc}}{T_1} + \Delta B) (1.5-h)2h^2 - \Delta B \right] \frac{1}{T_{acc}} \\
\ddot{q} = (\Delta C \frac{T_{acc}}{T_1} + \Delta B) (1-h) \frac{3h}{T_{acc}}
\end{cases}
\]

WHERE
\[
\begin{align*}
\Delta C &= C - B \\
\Delta B &= A - B \\
h &= \frac{t + T_{acc}}{2T_{acc}}
\end{align*}
\]

\[
q \rightarrow \{ JT \ COORD \ OR \ CARTESIAN \ COORD \}
\]
Can compute trajectory in real time; need only look ahead by one position

A. When \( t \geq t_a \) start calculating trajectory for path between \( t_c \) and \( t_d \), while moving between \( t_a \) and \( t_b \)... Eqns 1

B. Follow trajectory dictated by Eqns 2 between \( t_b \) and \( t_c \)

C. Implement trajectory calculated in step A and restart with parameters ----

\[
\begin{align*}
\frac{\dot{t}_a}{t_a} &= \frac{\Delta C}{T_1} \\
\frac{\dot{t}}{t} &= 0 \\
\Delta t &= \frac{t}{T_1}
\end{align*}
\]

calculated in step A.