Introduction to Spatial Mechanisms

At the end of this video, you should be able to:

• Describe the mobility of spatial mechanism joint types
• Determine the number of degrees-of-freedom of a spatial mechanism
• Describe the motion of a spherical mechanism

Examples of Spatial Mechanisms

Hooke (Cardan) Joint

Swashplate

Bone Cutter
Lower Pair Joint Types
Spatial Mechanism Naming Convention
Harrisberger’s “Most Useful” 1 DOF List
Mobility Defined by Kutzbach Criterion
Kutzback Criterion Example

\[ m = 6(n - 1) - 5j_1 - 4j_2 - 3j_3 - 2j_2 - j_1 \]

# Planar and Spatial Correlations

<table>
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<th>Planar Theory</th>
<th>Spherical Theory</th>
<th>Spatial Theory</th>
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<td>Kutzbach Criterion</td>
<td>Kutzbach Criterion</td>
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<td>Pole</td>
<td>Rotation Axis</td>
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<td>Instant Center</td>
<td>Instantaneous Rotation Screw</td>
<td>Instantaneous Screw</td>
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<td>Image Pole</td>
<td>Image Rotation Axis</td>
<td>Image Screw</td>
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<td>Pole Triangle</td>
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<td>Screw Triangle</td>
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<td>Center Point Curve</td>
<td>Fixed R-Axis Cone</td>
<td>Fixed C-Axis Congruent</td>
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<td>Moving R-Axis Cone</td>
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<td>5 Position R-R Axes</td>
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<td>Euler-Savary Equation</td>
<td>Spherical Euler-Savary Eqn</td>
<td>Spatial Euler-Savary Eqn</td>
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</table>
Spherical 4-bar
Spherical 4-bar
Spherical Mechanisms

\[ \alpha – \text{central angle of crank} \]
\[ \beta – \text{central angle of output} \]
\[ \gamma – \text{central angle of coupler} \]
\[ \zeta – \text{central angle of ground} \]
Spherical Coupler Curves

Spherical Roth-Burmester Curves
Representing Spatial Mechanisms: Denavit-Hartenberg Parameters

At the end of this video, you should be able to:

• Describe
DH Parameters

Conventions:
1. Number joints consecutively – start at the input
2. Choose joint axis such that: $x_i$ is perpendicular to $z_{i-1}$ and $z_i$
3. Origin $x_i, y_i, z_i$ is fixed in link w/ joints $i-1$ and $i$
4. In closed chain, joint 1 references last joint, joint N

4 D-H Parameters:
- $a_{i,i+1}$: dist along $x_{i+1}$ from $z_i$ to $z_{i+1}$
- $\alpha_{i,i+1}$: angle from $z_i$ to $z_{i+1}$ (as seen from $x_{i+1}$)
- $\theta_{i,i+1}$: angle from $x_i$ to $x_{i+1}$ (as seen from $z_i$)
- $s_{i,i+1}$: dist along $z_i$ from $x_i$ to $x_{i+1}$
DH Parameter Example

\[ a_{i,i+1}: \text{dist along } x_{i+i} \text{ from } z_i \text{ to } z_{i+1} \]

\[ \alpha_{i,i+1}: \text{angle from } z_i \text{ to } z_{i+1} \text{ (as seen from } x_{i+1} \text{)} \]

\[ \theta_{i,i+1}: \text{angle from } x_i \text{ to } x_{i+1} \text{ (as seen from } z_i \text{)} \]

\[ s_{i,i+1}: \text{dist along } z_i \text{ from } x_i \text{ to } x_{i+1} \]

<table>
<thead>
<tr>
<th>Joint</th>
<th>( a )</th>
<th>( \alpha )</th>
<th>( \theta )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a_{0,1} = )</td>
<td>( \alpha_{0,1} = )</td>
<td>( \theta_{0,1} = )</td>
<td>( s_{0,1} = )</td>
</tr>
<tr>
<td>1</td>
<td>( a_{1,2} = )</td>
<td>( \alpha_{1,2} = )</td>
<td>( \theta_{1,2} = )</td>
<td>( s_{1,2} = )</td>
</tr>
<tr>
<td>2</td>
<td>( a_{2,3} = )</td>
<td>( \alpha_{2,3} = )</td>
<td>( \theta_{2,3} = )</td>
<td>( s_{2,3} = )</td>
</tr>
</tbody>
</table>

Source: irobotkits.blogspot.com
**DH Parameters**

\[ T_{i,i+1} = \begin{bmatrix}
\cos \theta_{i,i+1} & -\cos \alpha_{i,i+1} \sin \theta_{i,i+1} & \sin \alpha_{i,i+1} \sin \theta_{i,i+1} & a_{i,i+1} \cos \theta_{i,i+1} \\
\sin \theta_{i,i+1} & \cos \alpha_{i,i+1} \cos \theta_{i,i+1} & -\sin \alpha_{i,i+1} \cos \theta_{i,i+1} & a_{i,i+1} \sin \theta_{i,i+1} \\
0 & \sin \alpha_{i,i+1} & \cos \alpha_{i,i+1} & s_{i,i+1} \\
0 & 0 & 0 & 1
\end{bmatrix} \]

- \( a_{i,i+1} \): dist along \( x_{i+1} \) from \( z_i \) to \( z_{i+1} \)
- \( \alpha_{i,i+1} \): angle from \( z_i \) to \( z_{i+1} \) (as seen from \( x_{i+1} \))
- \( \theta_{i,i+1} \): angle from \( x_i \) to \( x_{i+1} \) (as seen from \( z_i \))
- \( s_{i,i+1} \): dist along \( z_i \) from \( x_i \) to \( x_{i+1} \)
DH Parameters: Closed Chain

\[
T_{i,i+1} = \begin{bmatrix}
\cos \theta_{i,i+1} & -\cos \alpha_{i,i+1} \sin \theta_{i,i+1} & \sin \alpha_{i,i+1} \sin \theta_{i,i+1} & a_{i,i+1} \cos \theta_{i,i+1} \\
\sin \theta_{i,i+1} & \cos \alpha_{i,i+1} \cos \theta_{i,i+1} & -\sin \alpha_{i,i+1} \cos \theta_{i,i+1} & a_{i,i+1} \sin \theta_{i,i+1} \\
0 & \sin \alpha_{i,i+1} & \cos \alpha_{i,i+1} & s_{i,i+1} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
• Label the appropriate DH parameters
• Write the symbolic matrix transformation eqn
• Draw the appropriate DH parameters
• Write the symbolic matrix transformation eqn

\[ I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \begin{align*}
T_{12} & = \begin{pmatrix} a_1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
T_{23} & = \begin{pmatrix} a_2 & 0 & 0 & l_2 \\ 0 & c_{\theta_{23}} & -s_{\theta_{23}} & 0 \\ 0 & s_{\theta_{23}} & c_{\theta_{23}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
T_{34} & = \begin{pmatrix} a_3 & 0 & 0 & l_3 \\ 0 & c_{\theta_{34}} & -s_{\theta_{34}} & 0 \\ 0 & s_{\theta_{34}} & c_{\theta_{34}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
T_{41} & = \begin{pmatrix} a_4 & 0 & 0 & l_4 \\ 0 & c_{\theta_{41}} & -s_{\theta_{41}} & 0 \\ 0 & s_{\theta_{41}} & c_{\theta_{41}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*} \]
Spherical Four-Bar
1. Label the x-axes
2. Draw the appropriate DH parameters
3. Write the symbolic matrix transformation equation
Hooke (Cardan) Joint Analysis
Hooke / Cardan Joint
\[ T_{1,2} T_{2,3} T_{3,4} = T_{4,1}^{-1} \]

\[
\begin{bmatrix}
\cos \phi_1 \cos \phi_2 \cos \phi_3 \\
+ \sin \phi_1 \sin \phi_3 \\
\sin \phi_1 \cos \phi_2 \cos \phi_3 \\
- \cos \phi_1 \sin \phi_3 \\
\sin \phi_2 \cos \phi_3 \\
0
\end{bmatrix}
\begin{bmatrix}
\cos \phi_1 \sin \phi_2 \\
- \cos \phi_1 \cos \phi_3 \\
- \sin \phi_1 \cos \phi_3 \\
\sin \phi_1 \cos \phi_2 \sin \phi_3 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\cos \phi_4 \\
- \cos \beta \sin \phi_4 \\
\sin \beta \sin \phi_4 \\
0
\end{bmatrix}
\begin{bmatrix}
\sin \phi_4 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
T_{1,2} = \begin{bmatrix}
\cos \phi_1 & 0 & \sin \phi_1 & 0 \\
\sin \phi_1 & 0 & \cos \phi_1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
T_{2,3} = \begin{bmatrix}
\cos \phi_2 & 0 & \sin \phi_2 & 0 \\
\sin \phi_2 & 0 & \cos \phi_2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
T_{3,4} = \begin{bmatrix}
\cos \phi_3 & 0 & \sin \phi_3 & 0 \\
\sin \phi_3 & 0 & \cos \phi_3 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
T_{4,1} = \begin{bmatrix}
\cos \phi_4 & - \cos \beta \sin \phi_4 & \sin \beta \sin \phi_4 & 0 \\
\sin \phi_4 & \cos \beta \cos \phi_4 & - \sin \beta \cos \phi_4 & 0 \\
0 & \sin \beta & \cos \beta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Driveshaft Phasing