**Frequency Response of Linear Time Invariant Systems**

Complex Numbers: Recall that every complex number has a magnitude and a phase.

![Diagram of complex number](image)

Example: \( z = a + bj, \quad j = \sqrt{-1} \)

- \( a \) is called the real part of \( z \), \( a = \text{Re}(z) \)
- \( b \) is called the imaginary part of \( z \), \( b = \text{Im}(z) \)

Magnitude of \( z \): \( |z| = \sqrt{(a^2 + b^2)} = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} \)

Phase of \( z \): \( \angle z = \tan^{-1} \left( \frac{b}{a} \right) = \tan^{-1} \left( \frac{\text{Im}(z)}{\text{Re}(z)} \right) \)

Both the magnitude and phase of a complex number are real.

What is the steady state response of any LTI system for a sinusoidal input of frequency \( \omega \)?

Assume that the system is stable: All its poles have negative real parts.

For example:

\[
X(s) = \frac{1}{ms^2 + bs + k} F(s)
\]
If \( F(s) = \frac{\omega}{s^2 + \omega^2} \) (sinusoid)

Then \( X(s) = \frac{1}{ms^2 + bs + k} \cdot \frac{\omega}{s^2 + \omega^2} \)

After partial fraction expansion, and inverse Laplace transforms, we will find:

\[
x(t) = Ae^{-\zeta \omega_n t} \sin \omega_d t + Be^{-\zeta \omega_n t} \cos \omega_d t + C \sin \omega t + D \cos \omega t
\]

Amplitude of steady state oscillation = \( \sqrt{C^2 + D^2} \)

Thus, steady state oscillation is at the same frequency as the input frequency.

**Example:** Given

\[
Y(s) = \frac{3}{s + 4} U(s)
\]

If the input \( u(t) = \sin 2t \), find the steady state response \( y(t) \).

**Solution:** \( u(t) = \sin 2t \)

\[
\Rightarrow U(s) = \frac{2}{s^2 + 4}
\]

\[
Y(s) = \frac{3 \cdot 2}{s + 4 \cdot s^2 + 4} = \frac{A}{s + 4} + \frac{Bs + C}{s^2 + 4}
\]

\[A(s^2 + 4) + (Bs + C)(s + 4) = 6\]

\[s = -4 \Rightarrow A((-4)^2 + 4) = 6\]

\[A = \frac{6}{20} = \frac{3}{10}\]

\[As^2 + Bs^2 + 0 \Rightarrow B = -A = -\frac{3}{10}\]

\[4A + 4C = 6\]
\(4C = 6 - 4A = 6 - \frac{12}{10} = \frac{48}{10}\)

\[C = \frac{12}{10} = \frac{6}{5}\]

\[Y(s) = \frac{3}{10} \cdot \frac{1}{s + 4} - \frac{3}{10} \cdot \frac{s}{s^2 + 4} + \frac{6}{5} \cdot \frac{1}{s^2 + 4}\]

\[y(t) = \frac{3}{10} e^{-4t} - \frac{3}{10} \cos 2t + \frac{3}{5} \sin 2t\]

\(\frac{3}{10} e^{-4t} : \) Transient response \(\rightarrow 0\) as \(t \rightarrow \infty\)

\(-\frac{3}{10} \cos 2t + \frac{3}{5} \sin 2t : \) Steady state response

Amplitude of steady state response? Phase of steady state response?

\[-\frac{3}{10} \cos 2t + \frac{6}{10} \sin 2t = \frac{\sqrt{45}}{10} \left[ -\frac{3}{\sqrt{45}} \cos 2t + \frac{6}{\sqrt{45}} \sin 2t \right]\]

\[= \frac{\sqrt{45}}{10} [\sin \alpha \cos 2t + \cos \alpha \sin 2t] = \frac{\sqrt{45}}{10} \sin (2t + \alpha)\]

\(\sqrt{3^2 + 6^2} = \sqrt{45}, \quad \cos \alpha = \frac{6}{\sqrt{45}}, \quad \sin \alpha = -\frac{3}{\sqrt{45}}\)

Amplitude \(= \frac{\sqrt{45}}{10} = \frac{\sqrt{9.5}}{10} = \frac{3\sqrt{5}}{10}\)

Phase = \(\alpha = \tan^{-1} \left( -\frac{3}{6} \right) = \tan^{-1} \left( -\frac{1}{2} \right)\)

It turns out that the amplitude and phase are given by:

\(|y| = |G(j\omega)|_{\omega=2}\)

\(\angle y = \angle G(j\omega)_{\omega=2}\)

Check: \(G(s)\) in this example is
Let $s = j\omega$, $j = \sqrt{-1}$

\[
G(j\omega) = \frac{3}{j\omega + 4} = \frac{3}{j\omega + 4} \cdot j\omega - 4 = \frac{3(j\omega - 4)}{j^2\omega^2 - 16} = \frac{3(j\omega - 4)}{-\omega^2 - 16}
\]

\[
= \frac{3(4 - j\omega)}{\omega^2 + 16} = \frac{12}{\omega^2 + 16} - \frac{3j\omega}{\omega^2 + 16}
\]

\[
\omega = 2 \frac{\text{rad}}{s}, \quad G(j\omega) = \frac{12}{4 + 16} - \frac{6}{4 + 16}j = \frac{12}{20} - \frac{6}{20}j = \frac{3}{5} - \frac{3}{10}j
\]

\[
|G(j\omega)|_{\omega = 2} = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{3}{10}\right)^2} = \frac{1}{10} \sqrt{6^2 + 3^2} = \frac{\sqrt{45}}{10}
\]

\[
\angle G(j\omega)_{\omega = 2} = \tan^{-1}\left(\frac{-\frac{3}{10}}{\frac{3}{5}}\right) = \tan^{-1}\left(\frac{-\frac{1}{10}}{\frac{1}{5}}\right) = \tan^{-1}\left(-\frac{1}{2}\right)
\]

Thus, the amplitude and phase of the response turned out to be equal to $|G(j\omega)|_{\omega = 2}$ and $\angle G(j\omega)_{\omega = 2}$ respectively.

**Generalized Result:**

If $Y(s) = G(s)U(s)$, where $G(s)$ is stable with all poles having negative real parts, and $u(t) = \sin\omega t$, then

\[
y(t) = |G(j\omega)|\sin(\omega t + \angle G(j\omega))
\]
Calculating magnitude and phase of a complex number

\[ z = a + bj, \]

\[ |z| = \sqrt{a^2 + b^2}, \quad \angle z = \tan^{-1}\left( \frac{b}{a} \right) \]

\[ z = \frac{a + bj}{c + dj} = \frac{(a + bj)(c - dj)}{(c + dj)(c - dj)} = \frac{(a + bj)(c - dj)}{c^2 - d^2j^2} \]

\[ = \frac{ac - adj + bcj - bdj^2}{c^2 + d^2} = \frac{(ac + bd) + j(bc - bd)}{c^2 + d^2} \]

\[ |z| = \frac{1}{c^2 + d^2} \sqrt{(ac + bd)^2 + (bc - bd)^2} \]

\[ = \frac{1}{c^2 + d^2} \sqrt{a^2c^2 + b^2d^2 + 2acbd + b^2c^2 + a^2d^2 - 2abcd} \]

\[ = \frac{1}{c^2 + d^2} \sqrt{a^2(c^2 + d^2) + b^2(a^2 + d^2)} \]

\[ = \frac{\sqrt{c^2 + d^2} \sqrt{a^2 + d^2}}{c^2 + d^2} = \frac{\sqrt{a^2 + d^2}}{\sqrt{c^2 + d^2}} \]

Hence

\[ z = \frac{a + bj}{c + dj} \Rightarrow |z| = \frac{\sqrt{a^2 + d^2}}{\sqrt{c^2 + d^2}} = \frac{\text{Mag of num}}{\text{Mag of den}} \]

Likewise \( z = (a + bj)(c + dj) \)

\[ |z| = \sqrt{a^2 + d^2} \sqrt{c^2 + d^2} = (\text{Mag of first factor})(\text{Mag of second factor}) \]

For phase calculations,

\[ z = (a + bj)(c + dj) \]

\[ \angle z = \tan^{-1}\left( \frac{b}{a} \right) + \tan^{-1}\left( \frac{d}{c} \right) \]
\[ z = \frac{a + bj}{c + dj} \]

\[ \angle z = \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(\frac{d}{c}\right) \]

Thus, calculating the magnitude and phase of a transfer function is easy and can be calculated from the magnitude and phase of the individual factors.

**Sample example:**

Let \( Y(s) = G(s)U(s) \), with

\[ G(s) = \frac{9s + 14}{3(s^2 + 4s + 3)} \]

If \( u(t) = 6\cos2t \), find the steady state response \( y_{ss}(t) \).

**Solution:**

\[ G(j\omega) = \frac{9j\omega + 14}{3(j^2\omega^2 + 4j\omega + 3)} \]

\[ G(j\omega)|_{\omega=2} = \frac{18j + 14}{3(-2^2 + 8j + 3)} = \frac{18j + 14}{-3 + 24j} \]

\[ |G(j\omega)|_{\omega=2} = \frac{\sqrt{18^2 + 14^2}}{\sqrt{3^2 + 24^2}} = 0.9427 \]

\[ \angle G(j\omega)|_{\omega=2} = \tan^{-1}\left(\frac{18}{14}\right) - \tan^{-1}\left(\frac{24}{-3}\right) = -0.7853 \text{ radians} \]

\[ y_{ss}(t) = 0.9427(6)\cos(2t - 0.7853), \text{ or} \]

\[ y_{ss}(t) = 5.656\cos(2t - 0.7853) \]
**Frequency Response Plots or Bode Plots**

$|G(j\omega)|$ can be plotted as a function of $\omega$.

$\angle G(j\omega)$ can be plotted as a function of $\omega$.

Together these are called the frequency response plot of $G(s)$.

The $|G(j\omega)|$ plot describes the amplitude of steady state oscillations over all frequencies.

**Example:** Input Force $F(t)$ Output: $x(t)$

\[
m\ddot{x} + b\dot{x} + kx = F(t) \quad \ldots \ldots \ldots \ldots \quad 3
\]

Find the frequency response of this system.

Find the Laplace transform of $3$, using the initial conditions as zero.

\[
ms^2X(s) + bsX(s) + kX(s) = F(s)
\]

\[
\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \quad \ldots \ldots \ldots \ldots \quad 4
\]

\[
\frac{X(s)}{F(s)} = G(s) = \frac{1}{ms^2 + bs + k}
\]

\[
G(j\omega) = \frac{1}{m(j\omega)^2 + bj\omega + k} = \frac{1}{-m\omega^2 + bj\omega + k} = \frac{1}{(k - m\omega^2) + j(b\omega)}
\]
If $|G(j\omega)|$ is plotted as a function of $\omega$, the plot is as follows:

It turns out we just need to learn the $|G(j\omega)|$ Vs. $\omega$ plot for 5 different transfer functions (all 1st and 2nd order transfer functions).

— We would then be able to interpret and understand frequency response of all systems.

Likewise, we just need to learn the $\angle G(j\omega)$ Vs. $\omega$ plot for the same 5 transfer functions.

— We can then interpret the phase plot of all systems.

For the mass-spring-damper system under consideration,
\[ G(j\omega) = \frac{1}{(k - m\omega^2) + j(b\omega)} \]

Hence

\[ \angle G(j\omega) = -\tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right) \]

\[ \angle G(j\omega) \] provides the phase difference between \( F(t) \) and \( x(t) \) at various input frequencies \( \omega \).

**Agenda:** To learn the magnitude frequency response plots for the following transfer functions

1. \( G(s) = \frac{1}{s} \)
2. \( G(s) = s \)
3. \( G(s) = \frac{1}{Ts+1} \)
4. \( G(s) = Ts + 1 \)
5. \( G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \)

We will use these 5 magnitude frequency response plots to analyze a # of engineering problems.